Homework 20

Doppler effect

The frequency (color) of the light and the angle at which the light source is seen depend on the velocity of the light source with respect to the observer. Let us see how does it work. Imagine a light source (a star) which is moving along x axis at a velocity $\boldsymbol{V}$ with respect to some inertial reference frame xyz. The star emits the light with a wavelength of $\boldsymbol{\lambda}'$ and frequency $\boldsymbol{\nu}'$.in the xý′-plane. An observer, who moves with the star, sees the star at an angle $\boldsymbol{\Theta}'$ (Figure1).

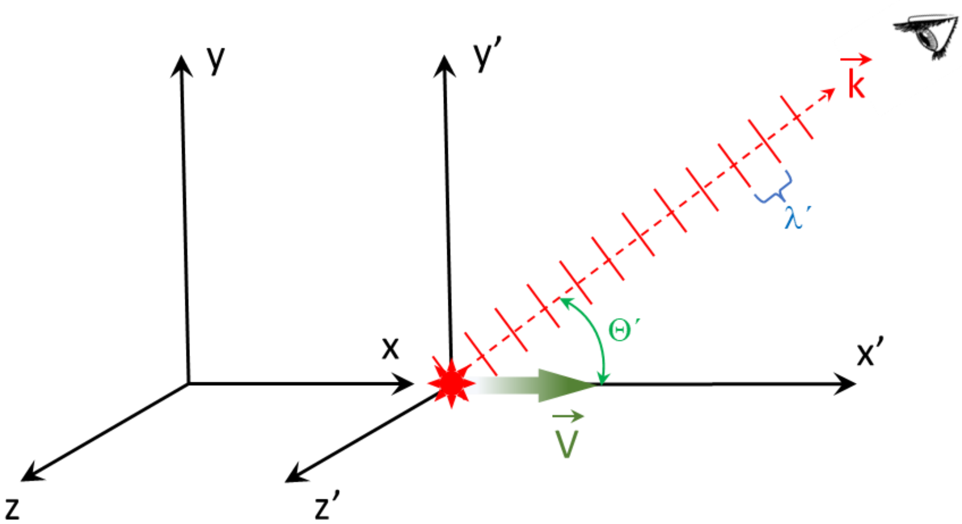

Figure 1.

The electric field of the light wave emitted by the star and propagating along the direction k, which forms angle $\boldsymbol{\Theta}'$ with the x′-axis in a reference frame x'y'z' moving together with the star can be written as:

$$E = E_0 cos\left[2\pi\left(\frac{cos\Theta\prime}{\lambda\prime}x' + \frac{sin\Theta\prime}{\lambda\prime}y' - \nu' t'\right)\right] \qquad (1).$$

Here $\boldsymbol{E_0}$ is the amplitude and $\boldsymbol{\lambda}'$ is the wavelength of the light wave, x' and y' – spatial coordinates and t' is time in the moving frame. We would like to what are the light frequency $\boldsymbol{\nu}$ and angle $\boldsymbol{\Theta}$ for an observer who is at rest, in the reference frame xyz. For this we have to replace x', y' and t' using Lorentz transformations:

$$\begin{cases} x' = \gamma(x - \beta ct) \\ ct' = \gamma(ct - \beta x) \end{cases} \qquad (2),$$

$$\text{here } \gamma = \frac{1}{\sqrt{1-\beta^2}}; \ \ \beta = \frac{V}{c} \qquad (3).$$

and $\boldsymbol{c}$ is the speed of light. Let us plug expressions (2) and (3) into (1). After distributing and rearranging the terms we have:

$$E = E_0 cos\left[2\pi\left(\gamma\left\{\frac{cos\Theta\prime}{\lambda\prime} + \nu'\frac{\beta}{c}\right\}x + \frac{sin\Theta\prime}{\lambda\prime}y - \gamma\left\{\frac{cos\Theta\prime}{\lambda\prime}c\beta + \nu'\right\}t\right)\right] \qquad (4).$$

Now we can recall that $\lambda' \cdot \nu' = \lambda \cdot \nu = c$. After using this in equation (4) we have:

$$E = E_0 cos\left[2\pi\left(\underbrace{\gamma\left\{\frac{cos\Theta' + \beta}{\lambda'}\right\}}_{\frac{\cos\Theta}{\lambda}}x + \underbrace{\frac{sin\Theta'}{\lambda'}}_{\frac{sin\Theta}{\lambda}}y - \underbrace{\gamma\nu'\{\beta cos\Theta' + 1\}t}_{\nu}\right)\right] \qquad (5).$$

From equation (5) we can conclude that angle Θ at which the observer at rest sees the moving star is different from that for the observer who is moving with the star (Θ'). From (5) we have:

$$tan\Theta = \frac{1}{\gamma} \cdot \frac{sin\Theta'}{cos\Theta' + \beta} \qquad (6).$$

Equation (6) is the *relativistic equation for the aberration of light*.

From equation (5) we see that the light frequency in the reference frame xyz (i.e. for the observer at rest) is:

$$\nu = \nu' \cdot \gamma(\beta cos\Theta' + 1) \qquad (7).$$

We can easily inverse this formula for the observer at rest: just to switch $\boldsymbol{\nu'}$ and $\boldsymbol{\nu}$; replace Θ' to Θ and $\boldsymbol{\beta}$ to $-\boldsymbol{\beta}$:

$$\nu' = \nu \cdot \gamma(1 - \beta cos\Theta) \qquad (8)$$

Substituting for $\boldsymbol{\gamma}$ and $\boldsymbol{\beta}$ we obtain:

$$\nu = \frac{\nu\prime \cdot \sqrt{1 - \frac{V^2}{c^2}}}{1 - \frac{V}{c}cos\Theta} \qquad (9)$$

If the star moves straight to the observer, then Θ=0, cosΘ=1 and the frequency is:

$$\nu = \nu' \cdot \sqrt{\frac{1 + \frac{V}{c}}{1 - \frac{V}{c}}} \qquad (10).$$

If the angle Θ=90º, cosΘ=0, then we still have red shift in the light frequency:

$$\nu = \nu' \sqrt{1 - \frac{V^2}{c^2}} \qquad (8)$$

This equation is called "*transverse Doppler effect*" and it is purely relativistic.

Problems:

1. How fast one should drive towards a traffic light to see the red light with a wavelength of 700nm blue shifter to green with the wavelength of 550nm?
2. Calculate the wavelength shift $\Delta\lambda=\lambda-\lambda_0$ for λ_0=589nm if the light source is moving at a velocity 0.1c and the angle $\boldsymbol{\Theta}$ between the line of sight and the velocity is
   a) 90º ?
   b) 85º ?

   Why does such a small departure from purely transverse motion generates such a large change in measured Doppler shift?