KINETIC ENERGY

JANUARY 22, 2024

Theory recap

We have already learned that for an isolated system of bodies (with no external forces acting on it) total momentum does not change with time. There is another conservation law you might have heard about: energy conservation law. Energy is a bit trickier than momentum because it has many forms and conservation only applies to the total energy, not individual forms - energy could change forms. Some of these forms are kinetic, potential and thermal energy, which will learn about in our course (some forms of energy will not be discussed in our course, such as chemical, electric or energy of light). Today we discuss the first type: kinetic energy.

We already know that an object of mass m moving with velocity v has momentum $p = mv$. Kinetic energy is another quantity which is associated with any moving object. Kinetic energy is equal to the mass of the object times its' speed squared and divided by 2:

$$E_{kinetic} = \frac{mv^2}{2}$$

Kinetic energy is a scalar quantity: it is just a number and it does not have a direction. A car moving to the right and a car moving to the left with the same speed will have the same kinetic energy (but not the same momentum).

There is a special name for the unit of energy: joule (after English physicist James Prescott Joule). Units of energy can be deduced from the formula for kinetic energy:

$$\mathrm{J} = \mathrm{kg} \cdot \frac{\mathrm{m}^2}{\mathrm{s}^2}$$

For example, consider an 80 kg cyclist moving with speed 5 m/s. What is cyclist's kinetic energy? $\frac{80\ \mathrm{kg} \cdot (5\ \mathrm{m/s})^2}{2} = 1000\ \mathrm{J} = 1\ \mathrm{kJ}$. Similarly to kilograms and kilometers 1000 joules is called a kilojoule, kJ.

If cyclist was moving with doubled speed, his kinetic energy would become four times larger, because kinetic energy is proportional to the square of speed. For kinetic energy speed is more important than mass, in contrast with momentum for which mass and speed are equally important.

Energy is very important because it is conserved. Very often energy is transferred between different forms, but there are situations in which only kinetic energy is important and it is separately conserved.

Let us look at a collision of two objects. During a collision, some of the initial kinetic energy of these objects can be lost to other forms. Many forms of dissipation can occur in a collision: heat and sound can be produced, there could be some structural damage that costs some energy of deformation. It is important that overall energy even in this case is conserved - kinetic energy gets transferred to other forms.

An example of collision with energy dissipation is an **inelastic collision**. Imagine that a ball hits another ball of the same mass which was initially at rest and the two balls stick together after the collision. What will be their common velocity v' in terms of the initial velocity of the first ball v? If the balls are moving without friction, we could use momentum conservation law, because there is no net external force acting on the system of two balls. Initial total momentum is mv since only one of the balls is moving. Final total momentum is $2mv'$, since both balls are moving together. From momentum conservation

$$mv = 2mv' \Longrightarrow v = 2v' \Longrightarrow v' = \frac{v}{2}.$$

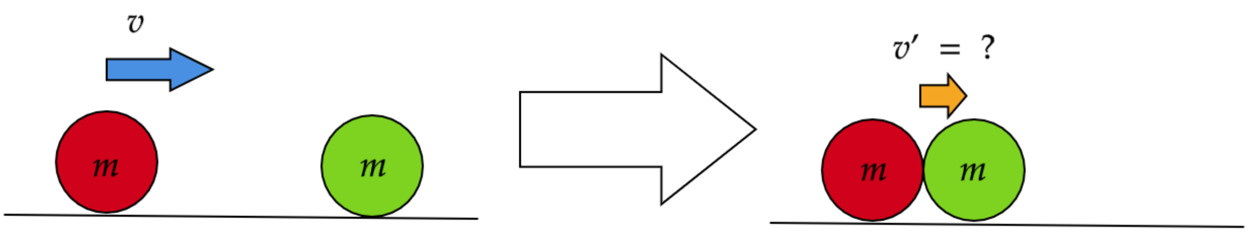

FIGURE 1. Inelastic collision of two balls

We see that as a result of the collision balls move together two times slower than the initial ball. Now let us find out what happens to the kinetic energy after the collision. Before the collision total kinetic energy is $\frac{mv^2}{2}$. After the collision total kinetic energy is

$$\frac{2m(v')^2}{2} = \frac{2m\left(\frac{v}{2}\right)^2}{2} = \frac{2m\frac{v^2}{4}}{2} = \frac{1}{2}\frac{mv^2}{2}$$

We see that only a half of initial kinetic energy remains in the form of kinetic energy after the collision, while the other half is dissipated into other forms of energy (heat, sound, etc.).

Another type of collision is a **completely elastic collision**. It is such a collision where dissipation is negligible, so that **kinetic energy is conserved**. For example, if you throw a rubber ball into a wall, it bounces elastically if its speed after hitting the wall is exactly the same as it was before the collision. Since speed is the same as before, kinetic energy is also the same.

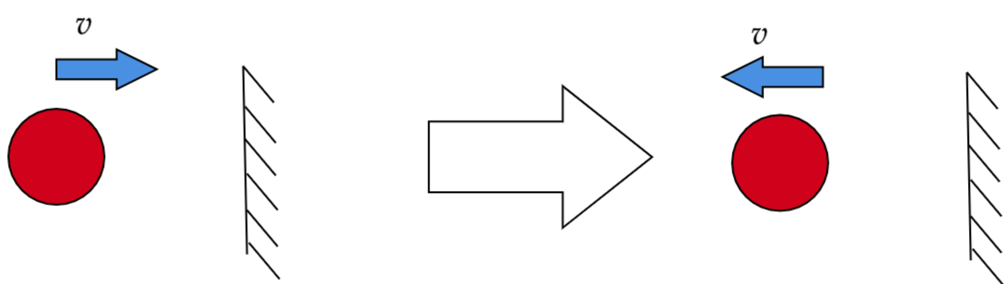

FIGURE 2. Completely elastic collision of a ball with a wall

The situation gets a lot more interesting if we have two balls. For example, this happens in the game of billiard. Let us find out, what happens if balls collide elastically. To set up the problem, consider two balls of the same mass m moving along a straight line without friction. For simplicity, assume that before collision the first ball moves with velocity v to the right and the second ball is at rest. Let us call their respective velocities after the collision v_1 and v_2, with positive values corresponding to the direction to the right.

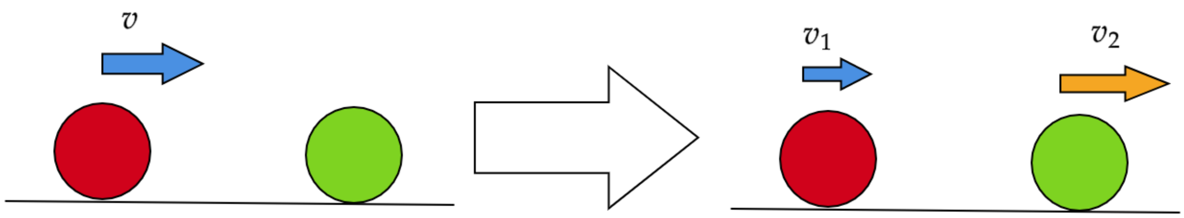

FIGURE 3. Completely elastic collision of two balls

Even before saying that the collision is completely elastic we could use momentum conservation law. The total momentum of the two balls is conserved because there are no external horizontal forces acting on them - there is no friction. Initial momentum of the first ball is mv and initial momentum of the second ball is 0, so the total initial momentum is mv. Final momentum of the first ball is mv_1 and final momentum of the second ball is mv_2, so the total final momentum is $mv_1 + mv_2$. Momentum conservation law thus tells us that

$$mv = mv_1 + mv_2 \Longrightarrow \boxed{v = v_1 + v_2}$$

Now let us use the fact that the collision is completely elastic. This means that total kinetic energy before the collision is equal to the total kinetic energy after the collision. Before the collision only one ball moves, so the total kinetic energy is $\frac{mv^2}{2} + 0 = \frac{mv^2}{2}$. After the collision kinetic energy of the first ball is $\frac{mv_1^2}{2}$ and kinetic energy of the second ball is $\frac{mv_2^2}{2}$, so the total kinetic energy is $\frac{mv_1^2}{2} + \frac{mv_2^2}{2}$. Energy conservation law reads:

$$\frac{mv^2}{2} = \frac{mv_1^2}{2} + \frac{mv_2^2}{2} \Longrightarrow \boxed{v^2 = v_1^2 + v_2^2}$$

Now we have a system of two boxed equations for the two unknowns (v_1 and v_2) and to solve them just requires maths. From the first equation we could express v_2: $v_2 = v - v_1$ and plug it into the second equation and do a series of simplifications:

$$v_1^2 + (v - v_1)^2 = v^2$$
$$v_1^2 + v^2 - 2vv_1 + v_1^2 = v^2$$
$$2v_1^2 - 2v_1 v = 2v_1(v - v_1) = 0 \Rightarrow \begin{cases} v_1 = 0 \text{ or} \\ v_1 = v \end{cases}$$

Recall that $v_2 = v - v_1$, so the full solutions are $v_1 = 0, v_2 = v$ and $v_1 = v, v_2 = 0$. We have obtained two answers, which one should we pick? Let us remember that the first ball is always to the left from the second ball and both v_1 and v_2 are directed to the right. Therefore it could not be that $v_1 > v_2$, because after the collision the first ball should fall behind the second one. So we choose $\boxed{v_1 = 0, v_2 = v}$.

We have obtained a very nice result: after the collision the balls just exchanged their velocities. The first ball comes to a stop while the second gets all of its speed. This is close to what one actually observes in billiard (which means that collisions in billiard are almost completely elastic). The result that after an elastic collision balls will exchange their velocities actually would hold for any initial velocities of the two balls of the same mass (if you want to know how to get this result, the easiest way would be to go to the reference frame in which one of the balls is at rest. Try to finish the proof by yourself, if you want).

You may wonder, why did we get two solutions? What does the discarded solution correspond to? The discarded solution $v_1 = v, v_2 = 0$ corresponds to the situation before the collision: the first ball moves and the second is at rest. This is indeed a solution of momentum conservation law and energy conservation law, because energy and momentum are of course conserved if velocities just stay what they are. But as discussed above, velocities can't stay the same after the collision happened, so we discard this solution.

Homework

1. Imagine that both the mass and the speed of a moving object increased 2 times. How did its kinetic energy change?

2. If two cars of different mass have the same momentum, which one has larger kinetic energy: a heavier car or a lighter car?

3. Calculate kinetic energy of a falling stone with a mass of 10 kg after 3 seconds of falling.

4. A runner moves with speed $v = 4$ m/s and has momentum $p = 250$ kg$\cdot$ m/s. Find kinetic energy of the runner. Derive a general formula for kinetic energy in terms of v and p.

***5.** 2023 identical balls are at rest, placed on a straight line at the interval of 1 m between the neighboring balls (see the figure below). 2024-th ball (the same as others) comes from the left with speed 1 m/s. How much time will pass between the first collision in this system and the last collision? How will all the balls move after the last collision? Size of the balls is much smaller than the distance between them.

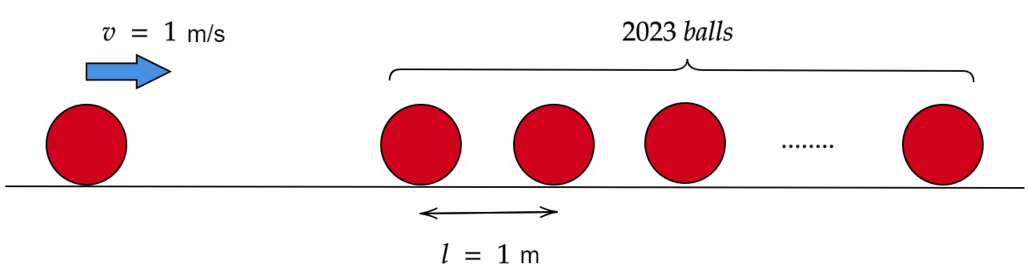

Figure 4. To problem *5