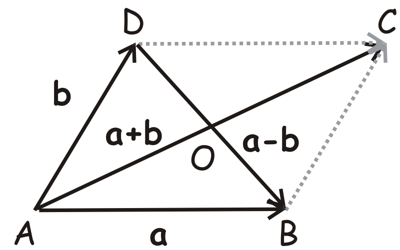
Geometry.

Solving vector problems.

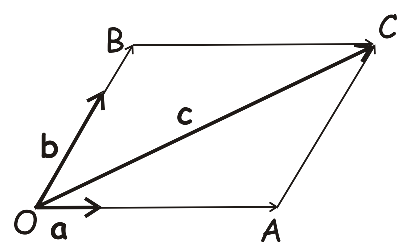
Problem. Prove that if vectors $\vec{a}$ and $\vec{b}$ satisfy $\|\vec{a}+\vec{b}\| = \|\vec{a}-\vec{b}\|$, then $\vec{a} \perp \vec{b}$.

Solution 1. Consider the vector addition parallelogram $ABCD$ shown in the Figure. Since its diagonals have equal length, $|AC| = \|\vec{a}+\vec{b}\| = \|\vec{a}-\vec{b}\| = |BD|$, the parallelogram is a rectangle (this is because the diagonals divide it into pairs of congruent isosceles triangles).



Solution 2. $(\vec{a}+\vec{b})^2 - (\vec{a}-\vec{b})^2 = 4(\vec{a}\cdot\vec{b}) = 4\,ab\cos\widehat{DAB} \Rightarrow \cos\widehat{DAB} = 0$ $\Rightarrow \widehat{DAB} = 90°$.

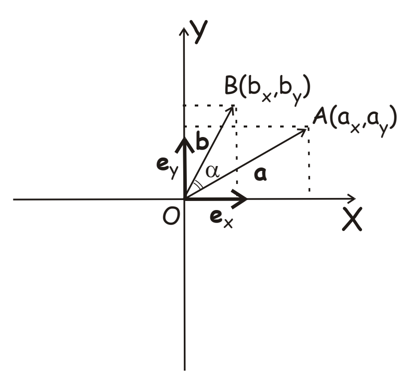
Problem. Show that for any two non-collinear vectors $\vec{a}$ and $\vec{b}$ in the plane and any third vector $\vec{c}$ in the plane, there exist one and only one pair of real numbers (x, y) such that $\vec{c}$ can be represented as $\vec{c} = x\vec{a} + y\vec{b}$.



Solution. Let us draw parallelogram $OACB$, whose diagonal is the segment OC, $\overrightarrow{OC} = \vec{c}$, and the sides OA and OB are parallel to the vectors $\vec{a}$ and $\vec{b}$, respectively. Since $\overrightarrow{OA} || \vec{a}$, there exists number x, such that $\overrightarrow{OA} = x \cdot \vec{a}$. Similarly, there exists number y, such that $\overrightarrow{OB} = y \cdot \vec{b}$. Then,

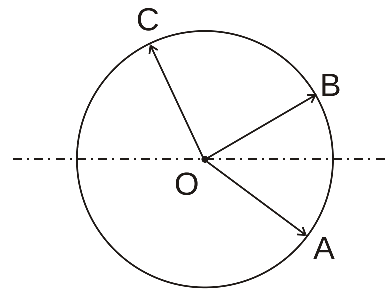
$$\overrightarrow{OC} = \overrightarrow{OA} + \overrightarrow{OB} = x \cdot \vec{a} + y \cdot \vec{b}.$$

Problem. Derive the formula for the scalar (dot) product of the two vectors, $\vec{a}(x_a, y_a)$ and $\vec{b}(x_b, y_b)$, $(\vec{a}\cdot\vec{b}) = x_a x_b + y_a y_b$, using their representation via two perpendicular vectors of unit length, $\vec{e}_x$ and $\vec{e}_y$, directed along the X and the Y axis, respectively.



Solution. It is clear from the Figure that $(\vec{a}\cdot\vec{b}) = ab\cos\alpha = ab\cos(\widehat{BOX} - \widehat{AOX}) = ab\cos\widehat{BOX}\cos\widehat{AOX} + ab\sin\widehat{BOX}\sin\widehat{AOX} = a_x b_x + a_y b_y$.

Problem. Vectors $\overrightarrow{OA}$, $\overrightarrow{OB}$ and $\overrightarrow{OC}$ are represented by the radial segments directed from the centre O of the circle to points A, b and C on the circle (see Figure). What are the angles $\widehat{AOB}$, $\widehat{AOC}$ and $\widehat{COB}$, if

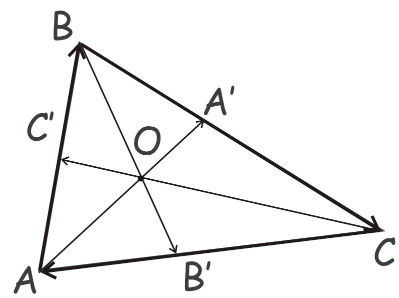


$$\overrightarrow{OC} = \overrightarrow{OA} - \overrightarrow{OB}$$

$$\overrightarrow{OC} = \overrightarrow{OA} + \overrightarrow{OB}$$

Solution. It is clear from the Figure that if $\overrightarrow{OC} = \overrightarrow{OB} - \overrightarrow{OA} = \overrightarrow{BA}$, then $\widehat{AOB} = \widehat{AOC} = 60°$ and $\widehat{COB} = 120°$. In the second case the situation looks similar to that in the figure, but with points B and C interchanged. Therefore, $\widehat{AOB} = 120°$ and $\widehat{AOC} = \widehat{COB} = 60°$.

Problem. Vectors $\overrightarrow{AA'}$, $\overrightarrow{BB'}$ and $\overrightarrow{CC'}$ are represented by the internal bisectors in the triangle ABC, directed from each vertex to the point on the opposite side (see figure). Express the sum, $\overrightarrow{AA'} + \overrightarrow{BB'} + \overrightarrow{CC'}$ through vectors $\overrightarrow{AB}$ and $\overrightarrow{AC}$ (and the sides of the triangle, $|AB| = c$, $|BC| = a$, $|CA| = b$). For what triangles ABC does this sum equal 0?



Problem. Given three vectors, $\vec{a}$, $\vec{b}$ and $\vec{c}$, show that vector $\vec{d} = (\vec{b}\cdot\vec{c})\vec{a} - (\vec{a}\cdot\vec{c})\vec{b}$ is perpendicular to $\vec{c}$.

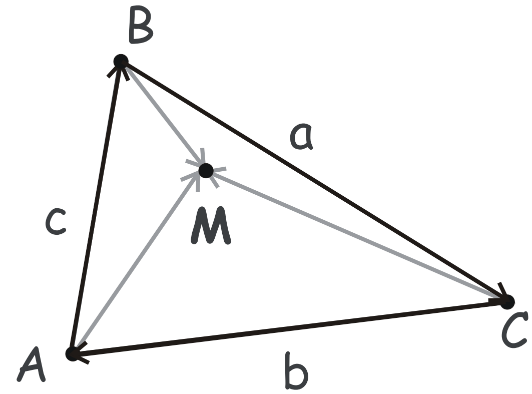
Solution. Let us find the scalar product $(\vec{d}\cdot\vec{c})$,

$$(\vec{d}\cdot\vec{c}) = \left(\left((\vec{b}\cdot\vec{c})\vec{a} - (\vec{a}\cdot\vec{c})\vec{b}\right)\cdot\vec{c}\right) = (\vec{b}\cdot\vec{c})(\vec{a}\cdot\vec{c}) - (\vec{a}\cdot\vec{c})(\vec{b}\cdot\vec{c}) = 0,$$

Which means that $\vec{d} = (\vec{b}\cdot\vec{c})\vec{a} - (\vec{a}\cdot\vec{c})\vec{b}$ is perpendicular to $\vec{c}$.

Problem. Given triangle ABC, find the locus of points M such that $(\overrightarrow{AB}\cdot\overrightarrow{CM}) + (\overrightarrow{BC}\cdot\overrightarrow{AM}) + (\overrightarrow{CA}\cdot\overrightarrow{BM}) = 0$. Using this finding, prove that three altitudes of the triangle ABC are concurrent (i.e. all three intersect at a common crossing point, the orthocenter of the triangle ABC).

Solution. Let M be an arbitrary point on the plane. Express (see Figure)



$$\overrightarrow{AB}=\overrightarrow{AM}-\overrightarrow{BM}, \overrightarrow{BC}=\overrightarrow{BM}-\overrightarrow{CM}, \overrightarrow{CA}=\overrightarrow{CM}-\overrightarrow{AM}.$$

Then, obviously,

$$\left(\overrightarrow{AB}\cdot\overrightarrow{CM}\right)+\left(\overrightarrow{BC}\cdot\overrightarrow{AM}\right)+\left(\overrightarrow{CA}\cdot\overrightarrow{BM}\right) = \left(\overrightarrow{AM}\cdot\overrightarrow{CM}\right)-\left(\overrightarrow{BM}\cdot\overrightarrow{CM}\right)+$$

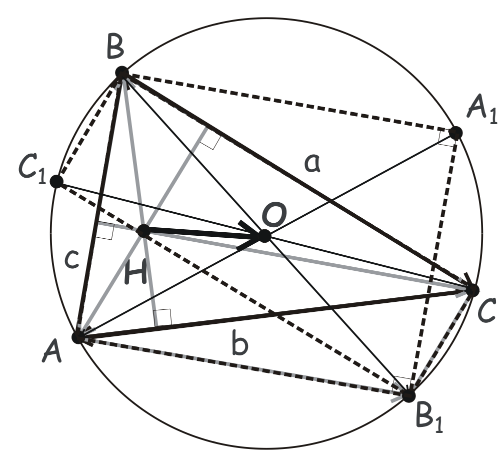
$$\left(\overrightarrow{BM}\cdot\overrightarrow{AM}\right)-\left(\overrightarrow{CM}\cdot\overrightarrow{AM}\right)+\left(\overrightarrow{CM}\cdot\overrightarrow{BM}\right)-\left(\overrightarrow{AM}\cdot\overrightarrow{BM}\right)=0$$

Hence, all points M on the plane satisfy the given vector condition.

Now, let H be the crossing point of the two altitudes of the triangle, $\overrightarrow{AA_1}$ and $\overrightarrow{BB_1}$. Then, $\left(\overrightarrow{BC}\cdot\overrightarrow{AH}\right)+\left(\overrightarrow{CA}\cdot\overrightarrow{BH}\right)=0$ by the definition of an altitude. However, we have just proved that for any point, H included, $\left(\overrightarrow{AB}\cdot\overrightarrow{CH}\right)+\left(\overrightarrow{BC}\cdot\overrightarrow{AH}\right)+\left(\overrightarrow{CA}\cdot\overrightarrow{BH}\right)=0$. Therefore, $\left(\overrightarrow{AB}\cdot\overrightarrow{CH}\right)=0$, and $\overrightarrow{CC_1}=\lambda\overrightarrow{CH}$ is also an altitude.

Problem. Let O be the circumcenter (a center of the circle circumscribed around) and H be the orthocenter (the intersection point of the three altitudes) of a triangle ABC. Prove that, $\overrightarrow{HA}+\overrightarrow{HB}+\overrightarrow{HC}=2\overrightarrow{HO}$.

Solution.



Let AA_1, BB_1 and CC_1, be the diameters of the circumcircle of the triangle ABC. Then, quadrilaterals ABA_1B_1 and BCB_1C_1 are rectangles (they are made of pairs of inscribed right triangles whose hypotenuse are the corresponding diameters), and $AHCB_1$ is parallelogram. Therefore, $\overrightarrow{HC}=\overrightarrow{AB_1}=\overrightarrow{BA_1}$ and $\overrightarrow{HA}=\overrightarrow{CB_1}=\overrightarrow{BC_1}$. Now, $\overrightarrow{HB}+\overrightarrow{HC}=\overrightarrow{HO}+\overrightarrow{OB}+\overrightarrow{HO}+\overrightarrow{OC}=2\overrightarrow{HO}+\overrightarrow{B_1O}+\overrightarrow{OC}=2\overrightarrow{HO}+\overrightarrow{B_1C}=2\overrightarrow{HO}-\overrightarrow{HA}$.

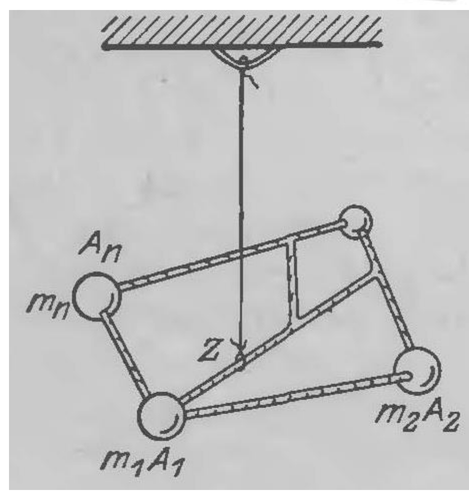
Vector approach to the Archimedes method of center of mass.

Let us assume that a system of geometric points, $X_1, X_2, X_3, \dots, X_n$ has masses $m_1, m_2, m_3, \dots, m_n$ associated with each point. The total mass of the system is $m = m_1 + m_2 + m_3 + \cdots + m_n$. By definition, the center of mass of such system is point M, such that

$$m_1 \cdot \overrightarrow{MX_1} + m_2 \cdot \overrightarrow{MX_2} + m_3 \cdot \overrightarrow{MX_3} + \cdots + m_n \cdot \overrightarrow{MX_n} = 0$$

For the case of just two massive points, $\{m_1, X_1\}$ and $\{m_2, X_2\}$ this reduces to $m_1 \cdot \overrightarrow{MX_1} = -m_2 \cdot \overrightarrow{MX_2}$, the Archimedes famous lever rule.

Heuristic properties of the Center of Mass.



1. Every system of finite number of point masses has unique center of mass (COM).
2. For two point masses, m_1 and m_2, the COM belongs to the segment connecting these points; its position is determined by the Archimedes lever rule: the point's mass times the distance from it to the COM is the same for both points, $m_1 d_1 = m_2 d_2$.
3. The position of the system's center of mass does not change if we move any subset of point masses in the system to the center of mass of this subset. In other words, we can replace any number of point masses with a single point mass, whose mass equals the sum of all these masses and which is positioned at their COM.

Given the coordinate system with the origin O, we can specify position of any geometric point A by the vector, $\overrightarrow{OA}$ connecting the origin O with this point. For the system of point masses, $m_1, m_2, m_3, \dots, m_n$, located at geometric points $X_1, X_2, X_3, \dots, X_n$, position of a point mass m_i is specified by the vector $\overrightarrow{OX_\iota}$ connecting the origin with point X_i where the mass is located.

It can be easily proven using the COM definition given above that the position of the COM, M, of the system is given by

$\overrightarrow{OM} = \frac{m_1 \cdot \overrightarrow{OX_1} + m_2 \cdot \overrightarrow{OX_2} + m_3 \cdot \overrightarrow{OX_3} + \cdots + m_n \cdot \overrightarrow{OX_n}}{m_1 + m_2 + m_3 + \cdots + m_n}$, or,

$$\overrightarrow{OM} = \frac{m_1 \cdot \overrightarrow{OX_1} + m_2 \cdot \overrightarrow{OX_2} + m_3 \cdot \overrightarrow{OX_3} + \cdots + m_n \cdot \overrightarrow{OX_n}}{m}$$

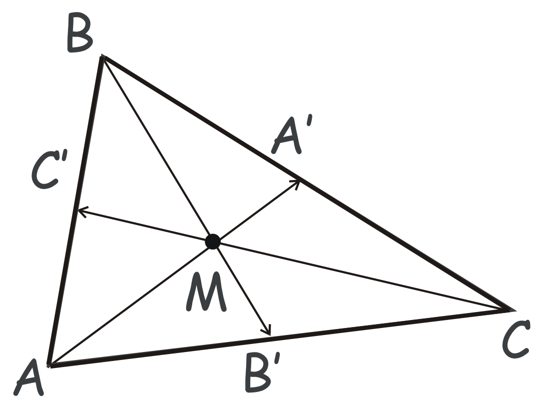
An important property of the COM immediately follows from the above. If we add a point (m_{n+1}, X_{n+1}) to the system, the resultant COM is the COM of the system of two points: the new point and the point (m, M) with mass m placed at the COM of the first n points,

$$\overrightarrow{OM}^{(n+1)} = \frac{m \cdot \overrightarrow{OM} + m_{n+1} \cdot \overrightarrow{OX_{n+1}}}{m + m_{n+1}}$$

$$\overrightarrow{OM}^{(n+1)} = \frac{m_1 \cdot \overrightarrow{OX_1} + m_2 \cdot \overrightarrow{OX_2} + m_3 \cdot \overrightarrow{OX_3} + \cdots + m_n \cdot \overrightarrow{OX_n} + m_{n+1} \cdot \overrightarrow{OX_{n+1}}}{m_1 + m_2 + m_3 + \cdots + m_n + m_{n+1}}$$

Problem. Prove that the medians of an arbitrary triangle ABC are concurrent (cross at the same point M).

Problem. Prove that the altitudes of an arbitrary triangle ABC are concurrent (cross at the same point H).



Problem. Prove that the bisectors of an arbitrary triangle ABC are concurrent (cross at the same point O).

Problem. Prove Ceva's theorem.