March 10, 2024 Math 9

Geometry.

Trigonometry homework review.

Solutions to selected homework problems:

1. Find the sum of the following series,

$$S_n = \cos x + \cos 2x + \cos 3x + \cos 4x + \cdots + \cos nx$$

(hint: multiply the sum by $2\sin x/2$)

Solution 1: Easy way of summing the trigonometric series is by multiplying and dividing it with $\sin\frac{x}{2}$,

$$S_n \frac{\sin\frac{x}{2}}{\sin\frac{x}{2}} = \frac{\sin\frac{x}{2}(\cos x+\cos 2x+\cdots+\cos nx)}{\sin\frac{x}{2}} = \frac{\sin\frac{x}{2}\cos x+\sin\frac{x}{2}\cos 2x+\cdots+\sin\frac{x}{2}\cos nx}{\sin\frac{x}{2}} =$$
$$\frac{\frac{1}{2}\left(-\sin\frac{x}{2}+\sin\frac{3x}{2}-\sin\frac{3x}{2}+\sin\frac{5x}{2}-\sin\frac{5x}{2}+\cdots-\sin\left(n-\frac{1}{2}\right)x+\sin\left(n+\frac{1}{2}\right)x\right)}{\sin\frac{x}{2}} = \frac{\frac{1}{2}\left(-\sin\frac{x}{2}+\sin\left(n+\frac{1}{2}\right)x\right)}{\sin\frac{x}{2}} =$$
$$\frac{\cos\frac{(n+1)x}{2}\sin\frac{nx}{2}}{\sin\frac{x}{2}}.$$

Solution 2. A slightly different and perhaps easier way of summing the above trigonometric series is by adding the expression for S_n, or S_2, rearranged from back to front, to itself, as we did when summing the arithmetic series,

$$S_n = \cos x + \cos 2x + \cdots + \cos nx$$

$$S_n = \cos nx + \cos(n-1)x + \cdots + \cos x$$

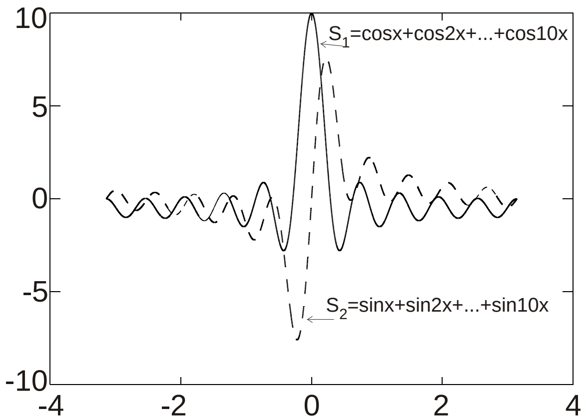
Wherefrom,

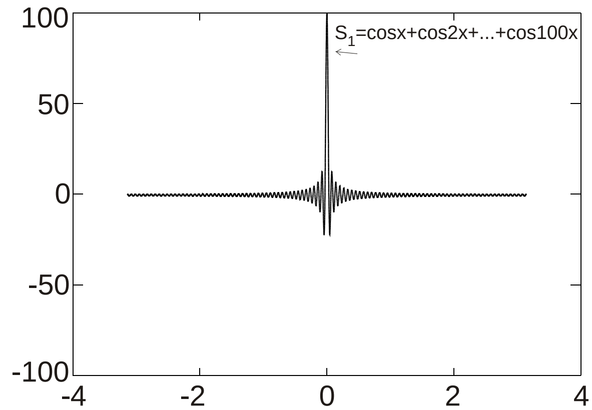
$$S_n = \frac{1}{2}\big((cos\, x + cos\, nx) + (cos\, 2x + cos(n-1)x) + \cdots + (cos\, nx +$$
$$cos\, x)\big) = cos\frac{(n+1)x}{2}\left(cos(n-1)\frac{x}{2} + cos(n-3)\frac{x}{2} + \cdots + cos(n-1)\frac{x}{2}\right) =$$

$$cos\,\frac{(n+1)x}{2}\,\frac{\left(sin\frac{x}{2}cos(n-1)\frac{x}{2}+sin\frac{x}{2}cos(n-3)\frac{x}{2}+\cdots+sin\frac{x}{2}\,cos(n-1)\frac{x}{2}\right)}{sin\frac{x}{2}} =$$

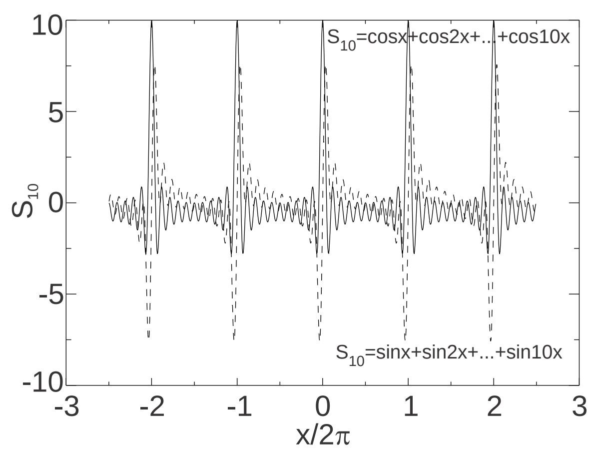
$$cos\,\frac{(n+1)x}{2}\,\frac{\frac{1}{2}\left(sin\frac{nx}{2}-sin\frac{(n-2)x}{2}+sin\frac{(n-2)x}{2}-sin\frac{(n-4)x}{2}+\cdots+sin\frac{nx}{2}\right)}{sin\frac{x}{2}} = cos\,\frac{(n+1)x}{2}\,\frac{sin\frac{nx}{2}}{sin\frac{x}{2}}.$$

It is interesting to look at a function $S_n(x)$.





Behavior of $S_n(x)$ is intuitively clear. For $x = 0$, all terms in the sum are equal to 1, and the sum equals to the number of terms, $S_n(0) = n$, while for $x \neq 0$ it consists of a large number of positive and negative terms, which tend to cancel each other. Note that is periodic with period 2π. The above figures only show one period. Here is how 5 periods look.



2. Prove the following equalities:

   a. $\frac{1}{\sin\alpha} + \frac{1}{\tan\alpha} = \cot\frac{\alpha}{2}$

Solution:

$$\frac{1}{\sin\alpha} + \frac{1}{\tan\alpha} = \frac{1+\cos\alpha}{\sin\alpha} = \frac{2\cos^2\frac{\alpha}{2}}{2\sin\frac{\alpha}{2}\cos\frac{\alpha}{2}} = \cot\frac{\alpha}{2}$$

b. $\sin^2\left(\frac{7\pi}{8}-2\alpha\right)-\sin^2\left(\frac{9\pi}{8}-2\alpha\right)=\frac{\sin 4\alpha}{\sqrt{2}}$

Solution:

$\sin^2\left(\frac{7\pi}{8}-2\alpha\right)-\sin^2\left(\frac{9\pi}{8}-2\alpha\right)=\left(\sin\left(\frac{7\pi}{8}-2\alpha\right)-\sin\left(\frac{9\pi}{8}-2\alpha\right)\right)\left(\sin\left(\frac{7\pi}{8}-2\alpha\right)+\sin\left(\frac{9\pi}{8}-2\alpha\right)\right)=2\cos\frac{2\pi-4\alpha}{2}\sin\left(-\frac{\pi}{8}\right)2\sin\frac{2\pi-4\alpha}{2}\cos\left(-\frac{\pi}{8}\right)=$
$-2\sin(\pi-2\alpha)\cos(\pi-2\alpha)\,2\sin\left(\frac{\pi}{8}\right)\cos\left(\frac{\pi}{8}\right)=\sin 4\alpha\sin\frac{\pi}{4}=\frac{\sin 4\alpha}{\sqrt{2}}$

c. $(\cos\alpha-\cos\beta)^2+(\sin\alpha-\sin\beta)^2=4\sin^2\frac{\alpha-\beta}{2}$

Solution:

$$\begin{aligned}(\cos\alpha-\cos\beta)^2&+(\sin\alpha-\sin\beta)^2\\ &=4\sin^2\frac{\alpha+\beta}{2}\sin^2\frac{\alpha-\beta}{2}+4\cos^2\frac{\alpha+\beta}{2}\sin^2\frac{\alpha-\beta}{2}\\ &=4\sin^2\frac{\alpha-\beta}{2}\left(\sin^2\frac{\alpha+\beta}{2}+\cos^2\frac{\alpha+\beta}{2}\right)=4\sin^2\frac{\alpha-\beta}{2}\end{aligned}$$

d. $\frac{\cot^2 2\alpha-1}{2\cot 2\alpha}-\cos 8\alpha\cot 4\alpha=\sin 8\alpha$

Solution:

$\frac{\cot^2 2\alpha-1}{2\cot 2\alpha}-\cos 8\alpha\cot 4\alpha=\frac{\cos^2 2\alpha-\sin^2 2\alpha}{2\sin 2\alpha\cos 2\alpha}-\cos 8\alpha\frac{\cos 4\alpha}{\sin 4\alpha}=\cot 4\alpha\,(1-\cos 8\alpha)=\frac{\cos 4\alpha}{\sin 4\alpha}2\sin^2 4\alpha=2\sin 4\alpha\cos 4\alpha=\sin 8\alpha$

e. $\sin^6\alpha+\cos^6\alpha+3\sin^2\alpha\cos^2\alpha=1$

Solution:

$\sin^6\alpha+\cos^6\alpha+3\sin^2\alpha\cos^2\alpha=\left(\frac{3\sin\alpha-\sin 3\alpha}{4}\right)^2+\left(\frac{\cos 3\alpha+3\cos\alpha}{4}\right)^2+\frac{3}{4}\sin^2 2\alpha=\frac{1}{16}(9\sin^2\alpha+\sin^2 3\alpha-6\sin\alpha\sin 3\alpha+\cos^2 3\alpha+9\cos^2\alpha+6\cos 3\alpha\cos\alpha+12\sin^2 2\alpha)=\frac{1}{16}\big(10+6(\cos 3\alpha\cos\alpha-\sin 3\alpha\sin\alpha)+6(1-\cos 4\alpha)\big)=\frac{1}{16}\big(10+6\cos 4\alpha+6(1-\cos 4\alpha)\big)=1$

f. $\frac{\sin 6\alpha+\sin 7\alpha+\sin 8\alpha+\sin 9\alpha}{\cos 6\alpha+\cos 7\alpha+\cos 8\alpha+\cos 9\alpha} = \tan\frac{15\alpha}{2}$

Solution:

$$\frac{\sin 6\alpha+\sin 7\alpha+\sin 8\alpha+\sin 9\alpha}{\cos 6\alpha+\cos 7\alpha+\cos 8\alpha+\cos 9\alpha} = \frac{(\sin 6\alpha+\sin 9\alpha)+(\sin 8\alpha+\sin 7\alpha)}{(\cos 6\alpha+\cos 9\alpha)+(\cos 8\alpha+\cos 7\alpha)} =$$
$$\frac{2\sin\frac{15}{2}\alpha\cos\frac{3}{2}\alpha+2\sin\frac{15}{2}\alpha\cos\frac{1}{2}\alpha}{2\cos\frac{15}{2}\alpha\cos\frac{3}{2}\alpha+2\cos\frac{15}{2}\alpha\cos\frac{1}{2}\alpha} = \frac{\sin\frac{15}{2}\alpha}{\cos\frac{15}{2}\alpha}\frac{\cos\frac{3}{2}\alpha+\cos\frac{1}{2}\alpha}{\cos\frac{3}{2}\alpha+\cos\frac{1}{2}\alpha} = \tan\frac{15}{2}\alpha$$

g. $\sin^6\alpha+\cos^6\alpha = \frac{5+3\cos 4\alpha}{8}$

Solution:

$$\sin^6\alpha+\cos^6\alpha = \left(\frac{3\sin\alpha-\sin 3\alpha}{4}\right)^2 + \left(\frac{\cos 3\alpha+3\cos\alpha}{4}\right)^2 = \frac{1}{16}(9\sin^2\alpha+\sin^2 3\alpha -$$
$$6\sin\alpha\sin 3\alpha+\cos^2 3\alpha+9\cos^2\alpha+6\cos 3\alpha\cos\alpha) = \frac{1}{16}\big(10+$$
$$6(\cos 3\alpha\cos\alpha-\sin 3\alpha\sin\alpha)\big) = \frac{1}{16}(10+6\cos 4\alpha) = \frac{5+3\cos 4\alpha}{8}$$

h. $16\sin^5\alpha-20\sin^3\alpha+5\sin\alpha = \sin 5\alpha$

Solution:

$$\sin 5\alpha = \sin\alpha\cos 4\alpha+\cos\alpha\sin 4\alpha = \sin\alpha\,(2\cos^2 2\alpha-1)+$$
$$\cos\alpha\, 2\sin 2\alpha\cos 2\alpha = \sin\alpha\,(2(1-2\sin^2\alpha)^2-1)+4\sin\alpha\cos^2\alpha\,(1-$$
$$2\sin^2\alpha) = \sin\alpha\,\big(1-8\sin^2\alpha+8\sin^4\alpha+4(1-\sin^2\alpha)(1-$$
$$2\sin^2\alpha)\big) = \sin\alpha\,(5-20\sin^2\alpha+16\sin^4\alpha)$$

i. $\frac{\cos 64°\cos 4°-\cos 86°\cos 26°}{\cos 71°\cos 41°-\cos 49°\cos 19°}$

Solution:

$$\frac{\cos 64°\cos 4°-\cos 86°\cos 26°}{\cos 71°\cos 41°-\cos 49°\cos 19°} = \frac{\cos 60°+\cos 68°-\cos 60°-\cos 112°}{\cos 30°+\cos 112°-\cos 30°-\cos 68°} = \frac{\cos 68°-\cos 112°}{\cos 112°-\cos 68°} = -1$$

j. $\sin 20°\sin 40°\sin 60°\sin 80° = \frac{3}{16}$

Solution: denote $x = 20°$, $\cos 3x = \cos 60° = \frac{1}{2}$, $\cos 6x = \cos 120° = -\frac{1}{2}$,

$\sin x \sin 2x \sin 3x \sin 4x = \sin x \sin 3x \sin 2x \sin 4x = \frac{1}{2}(\cos 2x - \cos 4x)\frac{1}{2}(\cos 2x - \cos 6x) = \frac{1}{2}\left(\cos 2x - (2\cos^2 2x - 1)\right)\left(\cos 2x + \frac{1}{2}\right) = \frac{1}{2}\left(\cos^2 2x - 2\cos^3 2x + \cos 2x + \frac{1}{2}\cos 2x - \cos^2 2x + \frac{1}{2}\right) = \frac{1}{4}(1 - 4\cos^3 2x + 3\cos 2x) = \frac{1}{4}(1 - \cos 6x) = \frac{3}{16}$

k. $\frac{1}{\sin 10°} - \frac{\sqrt{3}}{\cos 10°} = 4$

Solution:

$$\frac{1}{\sin 10°} - \frac{\sqrt{3}}{\cos 10°} = 4\frac{\frac{1}{2}\cos 10° - \frac{\sqrt{3}}{2}\sin 10°}{2\sin 10° \cos 10°} = 4\frac{\frac{1}{2}\cos 10° - \frac{\sqrt{3}}{2}\sin 10°}{2\sin 10° \cos 10°} = 4\frac{\sin(30° - 10°)}{\sin 20°} = 4$$

Trigonometry homework review. Part 2.

3. Simplify the following expressions:

l. $\sin^2\left(\frac{\alpha}{2} + 2\beta\right) - \sin^2\left(\frac{\alpha}{2} - 2\beta\right)$

Solution:

$\sin^2\left(\frac{\alpha}{2} + 2\beta\right) - \sin^2\left(\frac{\alpha}{2} - 2\beta\right) = \left(\sin\left(\frac{\alpha}{2} + 2\beta\right) - \sin\left(\frac{\alpha}{2} - 2\beta\right)\right)\left(\sin\left(\frac{\alpha}{2} + 2\beta\right) + \sin\left(\frac{\alpha}{2} - 2\beta\right)\right) = 2\cos\frac{\alpha}{2}\sin 2\beta \, 2\sin\frac{\alpha}{2}\cos 2\beta = \sin\alpha \sin 4\beta.$

m. $2\cos^2 3\alpha + \sqrt{3}\sin 6\alpha - 1$

Solution:

$$\begin{aligned} 2\cos^2 3\alpha + \sqrt{3}\sin 6\alpha - 1 &= \cos 6\alpha + 1 + \sqrt{3}\sin 6\alpha - 1 \\ &= 2\left(\frac{1}{2}\cos 6\alpha + \frac{\sqrt{3}}{2}\sin 6\alpha\right) = 2\left(\sin\frac{\pi}{6}\cos 6\alpha + \cos\frac{\pi}{6}\sin 6\alpha\right) \\ &= 2\sin\left(\frac{\pi}{6} + 6\alpha\right) \end{aligned}$$

n. $\cos^4 2\alpha - 6\cos^2 2\alpha \sin^2 2\alpha + \sin^4 2\alpha$

Solution:

$\cos^4 2\alpha - 6\cos^2 2\alpha \sin^2 2\alpha + \sin^4 2\alpha = (\cos^2 2\alpha - \sin^2 2\alpha)^2 - 4\cos^2 2\alpha \sin^2 2\alpha = \cos^2 4\alpha - \sin^2 4\alpha = \cos 8\alpha$

o. $\sin^2(135° - 2\alpha) - \sin^2(210° - 2\alpha) - \sin 195° \cos(165° - 4\alpha)$.

Solution:

$\sin^2(135° - 2\alpha) - \sin^2(210° - 2\alpha) - \sin\ 15° \cos(165° - 4\alpha) = \sin^2(45° + 2\alpha) - \sin^2(2\alpha - 30°) - \sin\ 15° \cos(15° + 4\alpha) = (\sin(45° + 2\alpha) - \sin(2\alpha - 30°))(\sin(45° + 2\alpha) + \sin(2\alpha - 30°)) - \sin\ 15° \cos(15° + 4\alpha) = 2\cos\frac{15°+4\alpha}{2}\sin\frac{75°}{2} 2\sin\frac{15°+4\alpha}{2}\cos\frac{75°}{2} - \sin\ 15° \cos(15° + 4\alpha) = \sin 75° \sin(15° + 4\alpha) - \sin\ 15° \cos(15° + 4\alpha) = \sin(15° + 4\alpha)\cos 15° - \cos(15° + 4\alpha)\sin\ 15° = \sin(4\alpha)$

p. $\frac{\cos 2\alpha - \cos 6\alpha + \cos 10\alpha - \cos 14\alpha}{\sin 2\alpha + \sin 6\alpha + \sin 10\alpha + \sin 14\alpha}$

Solution:

$$\frac{\cos 2\alpha - \cos 6\alpha + \cos 10\alpha - \cos 14\alpha}{\sin 2\alpha + \sin 6\alpha + \sin 10\alpha + \sin 14\alpha} = \frac{2\sin 8\alpha \sin 6\alpha - 2\sin 8\alpha \sin 2\alpha}{2\sin 8\alpha \cos 2\alpha + 2\sin 8\alpha \cos 6\alpha}$$
$$= \frac{\sin 6\alpha - \sin 2\alpha}{\cos 2\alpha + \cos 6\alpha} = \frac{2\cos 4\alpha \sin 2\alpha}{2\cos 4\alpha \cos 2\alpha} = \tan 2\alpha$$

4. Let A, B and C be angles of a triangle. Prove that

$$\tan\frac{A}{2}\tan\frac{B}{2} + \tan\frac{B}{2}\tan\frac{C}{2} + \tan\frac{C}{2}\tan\frac{A}{2} = 1$$

Solution:

$\tan\frac{A}{2}\tan\frac{B}{2} + \tan\frac{B}{2}\tan\frac{C}{2} + \tan\frac{C}{2}\tan\frac{A}{2} = \tan\frac{A}{2}\tan\frac{B}{2} + \tan\frac{B}{2}\tan\left(\frac{\pi}{2} - \frac{A+B}{2}\right) + \tan\left(\frac{\pi}{2} - \frac{A+B}{2}\right)\tan\frac{A}{2} = \tan\frac{A}{2}\tan\frac{B}{2} + \cot\frac{A+B}{2}\left(\tan\frac{A}{2} + \tan\frac{B}{2}\right) = \tan\frac{A}{2}\tan\frac{B}{2} + \cot\frac{A+B}{2}\frac{\sin\frac{A}{2}\cos\frac{B}{2} + \sin\frac{B}{2}\cos\frac{A}{2}}{\cos\frac{A}{2}\cos\frac{B}{2}} = \frac{\sin\frac{A}{2}\sin\frac{B}{2}}{\cos\frac{A}{2}\cos\frac{B}{2}} + \cot\frac{A+B}{2}\frac{\sin\frac{A+B}{2}}{\cos\frac{A}{2}\cos\frac{B}{2}} = \frac{\sin\frac{A}{2}\sin\frac{B}{2} + \cos\frac{A+B}{2}}{\cos\frac{A}{2}\cos\frac{B}{2}} = 1$