November 19, 2023

Algebra.

<u>Solutions to some homework problems</u>.

1. **Problem.** Write the first few terms in the following sequence ($n \geq 1$),

$$n\, fractions \begin{cases} \cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\cdots}}} \\ \qquad\qquad \ldots + \cfrac{1}{1+x} \end{cases} = f_n$$

   a. Try guessing the general formula of this fraction for any n.

   b. Using mathematical induction, try proving the formula you guessed.

Solution. $n = 1$: $f_1 = \frac{1}{1+x}$; $n = 2$: $f_2 = \frac{1}{1+\frac{1}{1+x}} = \frac{1+x}{2+x}$; $n = 3$, $f_3 = \frac{1}{1+\frac{1}{1+\frac{1}{1+x}}} = \frac{2+x}{3+2x}$; $n = 4$, $f_4 = \frac{1}{1+\frac{1}{1+\frac{1}{1+\frac{1}{1+x}}}} = \frac{3+2x}{5+3x}$; $f_5 = \frac{5+3x}{8+5x}$; … .

From the definition, we can write the recurrence, $f_{n+1} = \frac{1}{1+f_n}$. We note, that if $f_n = \frac{a_n+b_nx}{c_n+d_nx}$, then $f_{n+1} = \frac{c_n+d_nx}{(a_n+c_n)+(b_n+d_n)x}$. Hence, in each next term, f_{n+1}, in the sequence, the numerator is equal to the denominator of the previous term, f_n, while the numbers in the denominator are the sums of the corresponding numbers in the numerator and the denominator of the previous term, f_n, thus forming the Fibonacci sequence, $\{F_n\} = \{1,1,2,3,5,8,13, \ldots\}$. We can thus guess,

a. $n\, fractions$: $f_1 = \frac{1}{1+x}$, $f_n = \frac{F_n+F_{n-1}x}{F_{n+1}+F_nx}$, $n > 1$

b. <u>Base</u>: $f_2 = \frac{1+x}{1+2x}$

   <u>Induction</u>: Using the recurrence implied in the definition,

$$f_{n+1} = \frac{1}{1+f_n} = \frac{1}{1+\frac{F_n+F_{n-1}x}{F_{n+1}+F_nx}} = \frac{F_{n+1}+F_nx}{F_{n+1}+F_n+F_nx+F_{n-1}x} = \frac{F_{n+1}+F_nx}{F_{n+2}+F_{n+1}x}.$$

2. **Problem**. Can you prove that,

   a.

   $$\frac{3+\sqrt{17}}{2} = 3 + \cfrac{2}{3+\cfrac{2}{3+\cfrac{2}{3+\cdots}}}?$$

   b. $1 = 3 - \cfrac{2}{3-\cfrac{2}{3-\cfrac{2}{3-\cdots}}}?$

   c.

   $$\cfrac{4}{2+\cfrac{4}{2+\cfrac{4}{2+\cdots}}} = 1 + \cfrac{1}{4+\cfrac{1}{4+\cfrac{1}{4+\cdots}}}?$$

   Find these numbers?

Solution. Consider a general continued fraction,

$$x = a + \cfrac{b}{a+\cfrac{b}{a+\cfrac{b}{a+\cdots}}}$$

<u>If a number exists, which is equal to the above infinite continued fraction</u>, then it must satisfy the equation, $x = a + \frac{b}{x} \Leftrightarrow x^2 - ax - b = 0$ $\Leftrightarrow x = \frac{a}{2} \pm \sqrt{\left(\frac{a}{2}\right)^2 + b}$. If a and b are positive, then x must also be positive, so $x = \frac{a}{2} + \sqrt{\left(\frac{a}{2}\right)^2 + b}$.

a. Following the above argument with $a = 3$, $b = 2$, we obtain, $x = \frac{3}{2} + \sqrt{\left(\frac{3}{2}\right)^2 + 2} = \frac{3+\sqrt{17}}{2}$

b. In this case, $a = 3$, but $b = -2$ is negative. Applying the above considerations naively, we obtain, $x = 3 - \frac{2}{x} \Leftrightarrow x^2 - 3x + 2 = 0$ $\Leftrightarrow (x-1)(x-2) = 0$, i.e. there are two equally "legitimate" answers, $x = 1$, or $x = 2$. What this means, is that assumption that there exist unique number encoded by the given infinite continued fraction is wrong: there exists no such number! In fact, this can also be understood by looking at finite truncations approximating this continued fraction. If the continued fraction is truncated after subtracting 2 and before division by 3, then it is equal to 1,

$3 - \frac{2}{3-2} = 1, 3 - \frac{2}{3-\frac{2}{3-2}} = 1, \dots .$

If, on the other hand, the truncation is after division by 3 and before subtracting 2, then we obtain a sequence of numbers approaching 2,

$3 - \frac{2}{3} = 2\frac{1}{3}, 3 - \frac{2}{3-\frac{2}{3}} = 2\frac{1}{7}, 3 - \frac{2}{3-\frac{2}{3-\frac{2}{3}}} = 2\frac{1}{15}, \dots .$

c. Denote

$$x = \cfrac{4}{2 + \cfrac{4}{2 + \cfrac{4}{2 + \cdots}}} = \frac{4}{2+x}$$

Then, $x^2 + 2x - 4 = 0 \Leftrightarrow x = -1 \pm \frac{\sqrt{5}}{2}$, and $x > 0$. Hence, $x = -1 + \frac{\sqrt{5}}{2}$.

Similarly, denote

$$y = \cfrac{1}{4 + \cfrac{1}{4 + \cfrac{1}{4 + \cdots}}} = \frac{1}{4+y}$$

Then, $y^2 + 4y - 1 = 0 \Leftrightarrow y = -2 \pm \frac{\sqrt{5}}{2}$, and $y > 0$. Hence, $y = -2 + \frac{\sqrt{5}}{2}$, and $1 + y = -1 + \frac{\sqrt{5}}{2} = x$.