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Algebra.

Arithmetic and geometric mean inequality: Proof by induction.

The **arithmetic mean** of n numbers, $\{a_1, a_2, \dots, a_n\}$, is, by definition,

$$A_n = \frac{a_1+a_2+\cdots+a_n}{n} = \frac{1}{n}\sum_{i=1}^{n} a_i \tag{1}$$

The **geometric mean** of n non-negative numbers, $\{a_n \geq 0\}$, is, by definition,

$$G_n = \sqrt[n]{a_1 \cdot a_2 \cdot \dots \cdot a_n} = \sqrt[n]{\prod_{i=1}^{n} a_i} \tag{2}$$

Theorem. For any set of n non-negative numbers, the arithmetic mean is not smaller than the geometric mean,

$$\frac{a_1+a_2+\cdots+a_n}{n} \geq \sqrt[n]{a_1 \cdot a_2 \cdot \dots \cdot a_n} \tag{3}$$

The standard proof of this fact by mathematical induction is given below.

Induction basis. For $n = 1$ the statement is a true equality. We can also easily prove that it holds for $n = 2$. Indeed, $(a_1 + a_2)^2 - 4a_1a_2 = (a_1 - a_2)^2 \geq 0$ $\Rightarrow a_1 + a_2 \geq 2\sqrt{a_1a_2}$.

Induction hypothesis. Suppose the inequality holds for any set of n non-negative numbers, $\{a_1, a_2, \dots, a_n\}$.

Induction step. We must prove that the inequality then also holds for any set of $n + 1$ non-negative numbers, $\{a_1, a_2, \dots, a_{n+1}\}$.

Proof. If $a_1 = a_2 = \cdots = a_n = a_{n+1}$, then the equality, $A_{n+1} = G_{n+1}$, obviously holds. If not all numbers are equal, then there is the smallest (smaller than the mean) and the largest (larger than the mean). Let these be $a_{n+1} < A_{n+1}$, and $a_n > A_{n+1}$. Consider new sequence of n non-negative numbers, $\{a_1, a_2, \dots, a_{n-1}, a_n + a_{n+1} - A_{n+1}\}$. The arithmetic mean for these n numbers is still equal to A_{n+1},

$$\frac{a_1+a_2+\cdots+a_{n-1}+a_n+a_{n+1}-A_{n+1}}{n} = \frac{n+1}{n}A_{n+1} - \frac{1}{n}A_{n+1} = A_{n+1} \tag{4}$$

Therefore, by induction hypothesis,

$$(A_{n+1})^n \geq a_1 \cdot a_2 \cdot \ldots \cdot a_{n-1} \cdot (a_n + a_{n+1} - A_{n+1}) \tag{5}$$

$$(A_{n+1})^{n+1} \geq a_1 \cdot a_2 \cdot \ldots \cdot a_{n-1} \cdot (a_n + a_{n+1} - A_{n+1}) \cdot A_{n+1} \tag{6}$$

Wherein, using $a_{n+1} < A_{n+1}$ and $a_n > A_{n+1}$, as assumed above, we get $(a_n - A_{n+1})(A_{n+1} - a_{n+1}) > 0$, or, $a_n a_{n+1} < (a_n + a_{n+1} - A_{n+1})A_{n+1}$, so we could substitute the last two terms in the product with $a_n \cdot a_{n+1}$, while keeping the inequality. This completes the proof. ¤

Review of selected homework problems.

1. Using mathematical induction, prove that $\forall n \in \mathbb{N}$,
   a. $\sum_{k=1}^{n}(2k-1)^2 = 1^2 + 3^2 + 5^2 + \cdots + (2n-1)^2 = \frac{4n^3-n}{3}$,
   b. $\sum_{k=1}^{n}(2k)^2 = 2^2 + 4^2 + 6^2 + \cdots + (2n)^2 = \frac{2n(2n+1)(n+1)}{3}$
   c. $\sum_{k=1}^{n} k^3 = 1^3 + 2^3 + 3^3 + \cdots + n^3 = (1+2+3+\cdots+n)^2$
   d. $\sum_{k=1}^{n} \frac{1}{(2k-1)(2k+1)} = \frac{1}{1\cdot 3} + \frac{1}{3\cdot 5} + \frac{1}{5\cdot 7} + \cdots + \frac{1}{(2n-1)(2n+1)} < \frac{1}{2}$
   e. $\sum_{k=1}^{n} \frac{1}{(7k-6)(7k+1)} = \frac{1}{1\cdot 8} + \frac{1}{8\cdot 15} + \frac{1}{15\cdot 22} + \cdots + \frac{1}{(7n-6)(7n+1)} < \frac{1}{7}$
   f. $\sum_{k=n+1}^{3n+1} \frac{1}{k} = \frac{1}{n+1} + \frac{1}{n+2} + \frac{1}{n+3} + \cdots + \frac{1}{3n+1} > 1$

Solution of (f)

Basis: $P_1: \sum_{k=2}^{4} \frac{1}{k} = \frac{1}{2} + \frac{1}{3} + \frac{1}{4} > 1$

Induction: $P_n \Rightarrow P_{n+1}$, where $P_{n+1}: \sum_{k=n+2}^{3n+4} \frac{1}{k} = \frac{1}{n+2} + \frac{1}{n+3} + \cdots + \frac{1}{3n+4} > 1$

Proof: $\sum_{k=n+2}^{3n+4} \frac{1}{k} = \frac{1}{n+2} + \frac{1}{n+3} + \cdots + \frac{1}{3n+1} + \frac{1}{3n+2} + \frac{1}{3n+3} + \frac{1}{3n+4} = \sum_{k=n+1}^{3n+1} \frac{1}{k} + \frac{1}{3n+2} + \frac{1}{3n+3} + \frac{1}{3n+4} - \frac{1}{n+1} > 1$, because $\sum_{k=n+1}^{3n+1} \frac{1}{k} > 1$ by induction assumption, and $\frac{1}{3n+2} + \frac{1}{3n+3} + \frac{1}{3n+4} - \frac{1}{n+1} = \frac{1}{3}\left(\frac{1}{n+\frac{2}{3}} + \frac{1}{n+\frac{4}{3}} - \frac{2}{n+1}\right) = \frac{1}{3}\left(\frac{2n+2}{\left(n+\frac{2}{3}\right)\left(n+\frac{4}{3}\right)} - \frac{2}{n+1}\right) \geq \frac{1}{3}\left(\frac{2n+2}{(n+1)^2} - \frac{2}{n+1}\right) \geq 0$ (here we used the arithmetic-geometric mean inequality, $\sqrt{\left(n+\frac{2}{3}\right)\left(n+\frac{4}{3}\right)} \leq \frac{2n+2}{2} = n+1$).

2. Prove by mathematical induction that for any natural number n,
   a. $5^n + 6^n - 1$ is divisible by 10
   b. $9^{n+1} - 8n - 9$ is divisible by 64

Solution of (b)

<u>Basis</u>: P_1: $9^2 - 72 - 9 = 0$ is divisible by 64

<u>Induction</u>: $P_n \Rightarrow P_{n+1}$, where P_{n+1}: $\exists k \in \mathbb{Z}, 9^{n+2} - 8(n+1) - 9 = 64k$

<u>Proof</u>: $9^{n+2} - 8(n+1) - 9 = 9 \cdot 9^{n+1} - 8n - 17 = 9(9^{n+1} - 8n - 9) + 64n + 64 = 64k$ if P_n: $\exists k' \in \mathbb{Z}, 9^{n+1} - 8n - 9 = 64k'$

3. Problems on binomial coefficients, which are defined as,
   $C_n^k = {}_kC_n = \binom{n}{k} = \frac{n!}{k!(n-k)!}$.
   a. Prove that $C_{n+k}^2 + C_{n+k+1}^2$ is a full square
   b. Find n satisfying the following equation,

$$C_n^{n-1} + C_n^{n-2} + C_n^{n-3} + \cdots + C_n^{n-10} = 1023$$

   c. Prove that

$$\frac{C_n^1 + 2C_n^2 + 3C_n^3 + \cdots + nC_n^n}{n} = 2^{n-1}$$

Solution of (b)

$C_n^{n-1} + C_n^{n-2} + C_n^{n-3} + \cdots + C_n^{n-10} = C_n^1 + C_n^2 + C_n^3 + \cdots + C_n^{10} = C_n^0 + C_n^1 + C_n^2 + C_n^3 + \cdots + C_n^{10} - 1$, so, $C_n^0 + C_n^1 + C_n^2 + C_n^3 + \cdots + C_n^{10} = 1024 = 2^{10}$, which is satisfied for $n = 10$ thanks to the property of the binomial coefficients,

$$C_n^0 + C_n^1 + C_n^2 + \cdots + C_n^k + \cdots + C_n^n = (1+1)^n = 2^n$$

Solution of (c)

$$\frac{C_n^1 + 2C_n^2 + 3C_n^3 + \cdots + nC_n^n}{n} = C_{n-1}^0 + C_{n-1}^1 + C_{n-1}^2 + \cdots + C_{n-1}^{n-1} = 2^{n-1}$$