Classwork: Coordinate geometry review

1. Coordinate geometry: Introduction

The **midpoint** M of a segment AB with endpoints A(x1, y1) and B(x2, y2) has coordinates:

$$M\left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}\right)$$

The distance between two points $A(x1, y1)$ and $B(x2, y2)$ is given by the following formula:

$$\boldsymbol{d} = \sqrt{(\boldsymbol{x_2} - \boldsymbol{x_2})^2 + (\boldsymbol{y_2} - \boldsymbol{y_1})^2}$$

Line: $y = \boldsymbol{m}x + \boldsymbol{b}$

with a **slope** $m = \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1}$ and intercept ***b.***

Parabola: $y = ax^2 + bx + c$ (standard form) or $y = a(x-h)^2 + \mathrm{k}$ (vertex form)

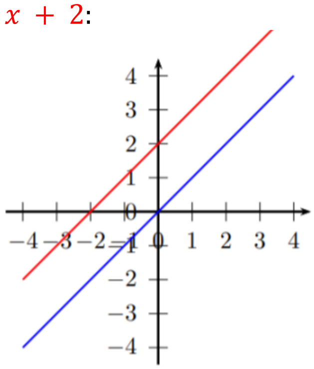
Circles: The equation of the circle with the center $M(x0, y0)$ and radius ***r*** is:$(x - x0)^2 + (y - y0)^2 = r^2$.

2. Graphs of functions and Function transformations

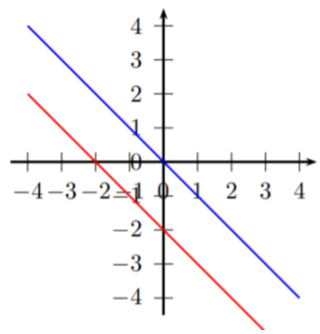
In general, the relation between x and y could be more complicated and could be given by some formula of the form $y = f(x)$, where f is some function of x (i.e., some formula which contains x). Then the set of all points whose coordinates satisfy this relation is called the **graph** of f.

Line. The graph of the function $y = mx + b$ is a straight line. The coefficient ***m*** is called the *slope*.

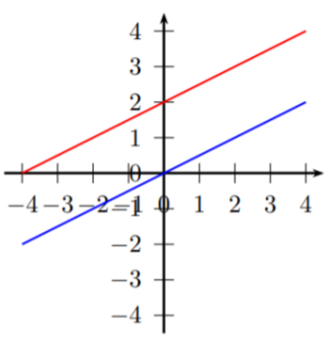
$y = x$; $y = x + 2$:



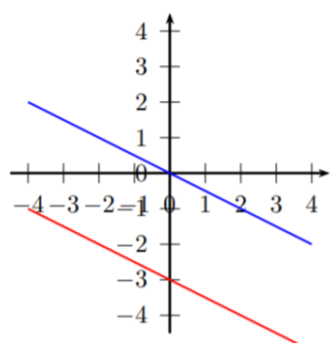
$y = -x$; $y = -x - 2$:



$y = \frac{1}{2}x$; $y = \frac{1}{2}x + 2$:

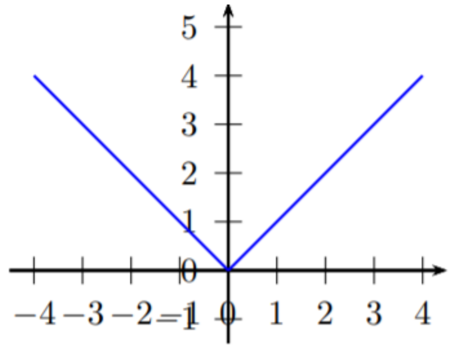


$y = -\frac{1}{2}x$; $y = -\frac{1}{2}x - 3$:



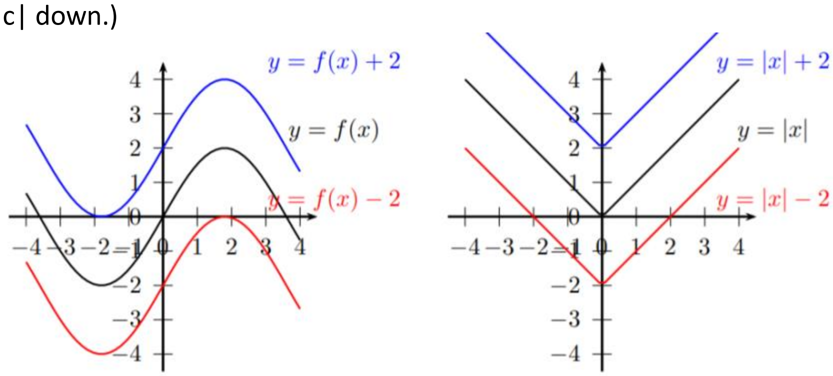
Absolute value of a line. $\boldsymbol{y = |x|}$

Two perpendicular lines, $y = x$ for $x > 0$ and $y = -x$ for $x < 0$.

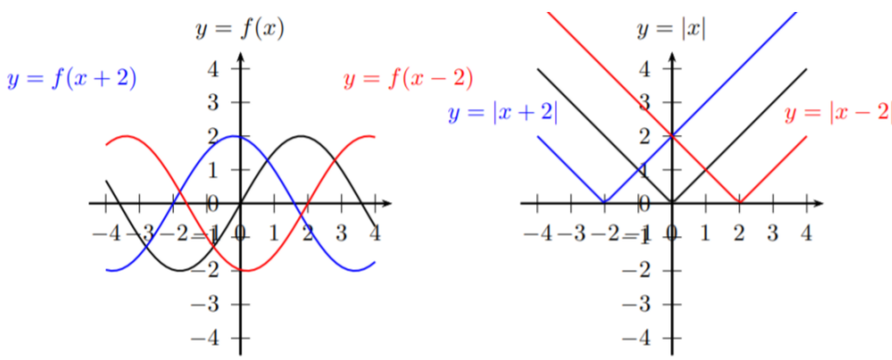


Having learned a number of basic graphs, we can produce new graphs, by doing certain transformations of the equations. Here are two of them.

Vertical translations: Adding constant ***c*** to the right-hand side of equation shifts the graph by ***c*** units up (if c is positive; if c is negative, it shifts by |c| down.)



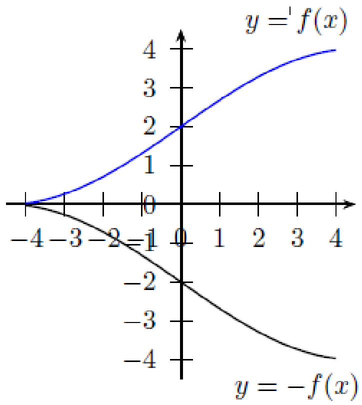
Horizontal translations: Adding constant ***c*** to x shifts the graph by ***c*** units left if ***c*** is positive; if ***c*** is negative, it shifts by ***c*** right.

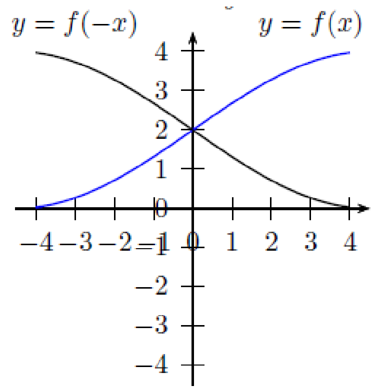


Reflection

Multiplying the function by −1 reflects the graph around the x-axis:

Replacing in the equation x by −x reflects the graph around the y-axis:





Graphs of functions (Summary)

In general, the relation between x and y could be more complicated and could be given by some formula of the form $y = f(x)$, where f is some function of x (i.e., some formula which contains x). Then the set of all points whose coordinates satisfy this relation is called the **graph** of f.

- Vertical shift: $y = f(x) + \boldsymbol{k}$, shift up by k $y = f(x) - \boldsymbol{k}$, shift down by k $k > 0$
- Horizontal shift: $y = f(x-h)$, shift right by h $y = f(x+h)$, shift to the left by h $h > 0$
- Reflection, x-axis: $y = -f(x)$, multiply the function by -1.
- Reflection, y-axis: $y = f(-x)$, multiply the argument by -1.

3. Quadratic function

Quadratic equation in a standard form: $ax^2 + bx + c = 0$

- a, b, c – coefficients, determinant D: $\boldsymbol{D = b^2 - 4ac}$, solutions(roots): $\boldsymbol{x_{1,2} = \frac{-b \pm \sqrt{D}}{2a}}$
- **D** determines the number of roots! ($D < 0$ no solutions, $D = 0$ one solution, $D > 0$ two solutions)

Quadratic function in a factored form: $y = a(x - x_1)(x - x_2)$, where

- roots: the numbers x_1 and x_2 – solutions of the quadratic equation ($y = 0$)
- **Vieta's formulas:** The roots are related to the coefficients: $\boldsymbol{x_1 x_2 = \frac{c}{a}}$ and $\boldsymbol{x_1 + x_2 = -\frac{b}{a}}$

Quadratic function in a vertex form: $\boldsymbol{y = a(x - h)^2 + k}$

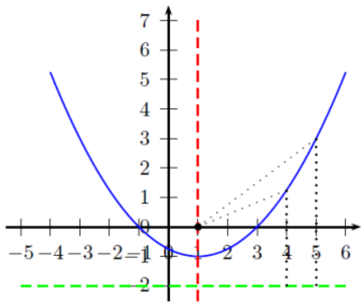
- **Method 1: completing the square**. Use the formulas for fast multiplication.
- **Method 2: find the vertex**. Determine the coefficients a, b, c. Find the vertex x-and y- coordinates

  $\boldsymbol{x_v = h = -\frac{b}{2a}}$. $\boldsymbol{y_v = k} = y(\boldsymbol{x_v}) = a\boldsymbol{x_v}^2 + b\boldsymbol{x_v} + c$

Modified vertex form: rewrite the equation into separate $y -$ and $x -$ part $4\boldsymbol{p}(y - \boldsymbol{k}) = (x - \boldsymbol{h})^2$

Distance from any point on the parabola to focus and directrix: $\boldsymbol{p} = \frac{1}{4a}$

Vertex $V(h, k)$ Focus $F(h, k + \boldsymbol{p})$ directrix $y = k - \boldsymbol{p}$



Parabola is the set of all points in a plane that are equally distant away from a given point and a given line (see black dotted lines). This given point is called **the focus** (black dot) of the parabola and the line is called **the directrix** (green line).

- If the parabola is of the form $(x-h)^2 = 4p(y-k)$, the vertex is $(h.k)$, the focus is $(h, k+p)$ and directrix is $y = k-p$.

Classwork problems

1. What is the equation of a line passing through point (2,1) and having slope ¾.
2. Write equation of a line passing through point (4,4) and parallel to line y=7/4 x -4 (Hint: parallel lines have same slope)
3. Write equation of a line) passing through point (5,-1) and perpendicular to line y=-5/2 x+5. (Hint: the product of slopes of two perpendicular lines is -1)

4. Find the intersection point of a line $y = x - 3$ and line $y = -2x + 6$.
5. Sketch the graphs of the following functions. Then, shift each function 2 units up, 2 units right, and reflect with respect to the x-axis. Sketch the result and write the equation.
    a. $y = |x+1|$
    b. $y = 1/x$
    c. $y = \sqrt{x}$
6. Find the coordinates of the points where the circle $(x+2)^2+(y-4)^2=5$ meets the line $y=-2x+4$.

7. Consider the circle $(x-5)^2+(y+2)^2=8$ and the lines $L1{:}y=-x-1$ and $L2{:}y=x-7$. How, if at all, do $L1$ and $L2$ intersect the circle?

8. Convert to vertex form and sketch the following graphs. Show the vertex point, the focus point, and the directrix line. You will have to calculate their coordinates/equations first.
    1) $y = -x^2 + 3x - 0.5$
    2) $y = x^2 + 4x - 4$

9. Prove that for any point P on the parabola y = x^2+1, the distance from P to the x-axis is equal to the distance from P to the point (0, 2).
10. A triangle ABC has corners A(−3, 0), B(0, 3) and (3, 0). The line y = 1/3 x + 1 separates the triangle in 2. What is the area of the piece lying below the line?