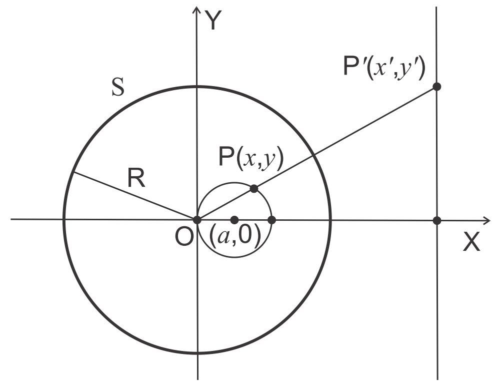
April 21, 2024 Math 9

Geometry.

Inversion in coordinate plane.

Consider inversion with respect to circle S centered at the origin, (0,0). Image of point $P(x, y)$ is point $P'(x', y')$. It is easy to see that the transformation of coordinates is (see figure),



$$x' = x\frac{R^2}{x^2 + y^2}$$
$$y' = y\frac{R^2}{x^2 + y^2}$$

And the inverse transformation,

$$x = x'\frac{R^2}{x'^2 + y'^2}$$
$$y = y'\frac{R^2}{x'^2 + y'^2}$$

The image of the line $y = ax$ is the line $y' = ax'$. Consider the image of the circle,

$$(x - a)^2 + y^2 = r^2$$

For points $P(x, y)$ on the circle, $x^2 + y^2 = r^2 - a^2 + 2ax$, so we have,

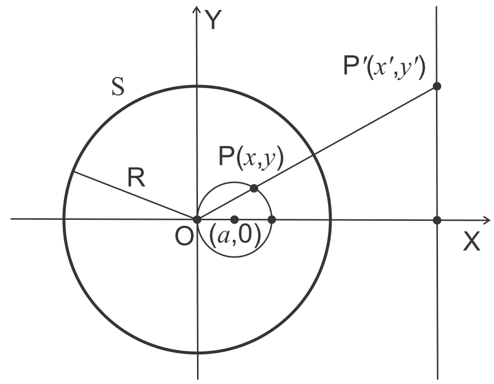
$$x' = x\frac{R^2}{r^2 - a^2 + 2ax}$$
$$y' = y\frac{R^2}{r^2 - a^2 + 2ax}$$

In the case where $a = r$, i.e. circle passes through the center of inversion, the image is a line,

$$x' = \frac{R^2}{2a}$$

Problem. Show that in the case $a \neq r$ there exist x_0, y_0, r_0, such that the image of circle $(x-a)^2 + y^2 = r^2$ is circle $(x'-x_0)^2 + (y'-y_0)^2 = r_0^2$.

Solution. Under an inversion with respect to a circle S centered at the origin, (0,0), the image of point $P(x,y)$ is point $P'(x',y')$. The transformation of the coordinates is (see figure),



$$x' = x\frac{R^2}{x^2+y^2}$$
$$y' = y\frac{R^2}{x^2+y^2}$$

Hence,

$$x'^2 + y'^2 = \frac{R^4}{x^2+y^2}$$

and

$$x = x'\frac{R^2}{x'^2+y'^2}$$
$$y = y'\frac{R^2}{x'^2+y'^2}$$

We thus have,

$$(x-a)^2 + y^2 = r^2 \Leftrightarrow x^2 + y^2 - 2ax + a^2 = r^2 \Leftrightarrow$$

$$\frac{R^4}{x'^2+y'^2} - 2ax'\frac{R^2}{x'^2+y'^2} = r^2 - a^2 \Leftrightarrow$$

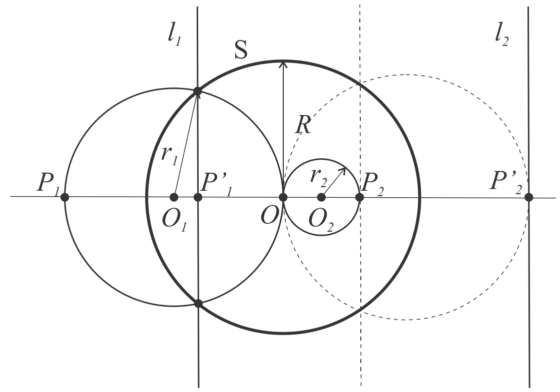
$$R^4 - 2ax'R^2 = \left(x'^2 + y'^2\right)(r^2 - a^2) \Leftrightarrow$$

$$x'^2 + 2ax'\frac{R^2}{r^2-a^2} + y'^2 = \frac{R^4}{r^2-a^2} \Leftrightarrow$$

$$\left(x' + \frac{aR^2}{r^2-a^2}\right)^2 + y'^2 = \frac{R^4}{r^2-a^2} + a^2\frac{R^4}{(r^2-a^2)^2} = \frac{r^2R^4}{(r^2-a^2)^2}$$

Wherefrom we find, $x_0 = \frac{aR^2}{a^2-r^2}$ and $r_0 = \left|\frac{rR^2}{r^2-a^2}\right|$.

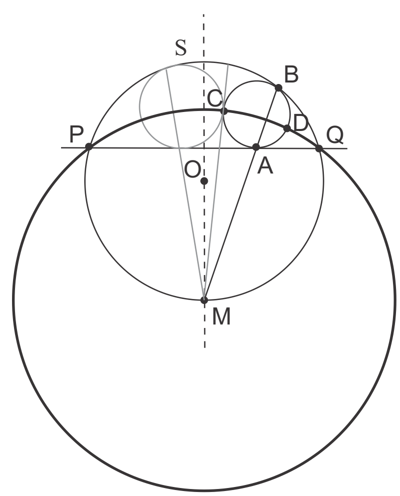
Inversive geometry. Homework review.



Problem. Find the distance between two parallel straight lines that are images of the two circles with the radii r_1 and r_2, which are tangent at the center O of the inversion circle S with radius R.

Solution. Consider the figure. Under the inversion in the circle (O, R), line l_1 is the image of the circle (O_1, r_1) and line l_2 is the image of the circle (O_2, r_2). The distance between the two lines, $|P'_1P'_2| = |OP'_1| + |OP'_2| = \frac{R^2}{|OP_1|} + \frac{R^2}{|OP_2|} = \frac{R^2}{2r_1} + \frac{R^2}{2r_2}$. Here, the pre-images of P'_1 and P'_2 are the points P_1 and P_2, which are intersections of the circles (O_1, r_1) and (O_2, r_2) with the line O_1O_2 connecting their centers and also passing through the inversion center, O.

Problem. Consider a circle S with center O and a straight line PQ that cuts from S a circular segment PSQ.

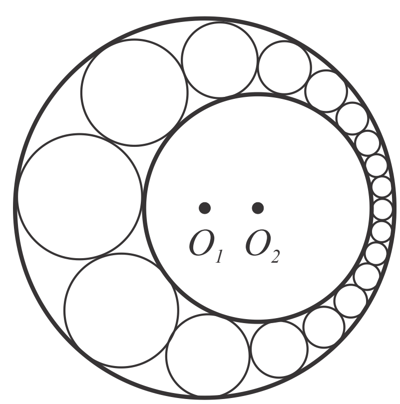


a. Prove that for any circle inscribed in the segment the line joining the tangency points A and B with the segment and with the circle passes through the midpoint M of the arc PMQ complementary to the segment.
b. Prove that if two circles inscribed in a circular segment PSQ touch, their common tangent passes through M.
c. Prove that if two circles inscribed in a circular segment PSQ cross, the line through the two points of intersection passes through M.

d. A circle overlaps a circular segment so that the four angles it forms with the boundary of the segment are all equal. Let the points of intersection be A_1 and A_2 on the linear segment and B_1 and B_2 on the arc such that A_1B_2 intersect A_2B_1 inside the segment. Then A_1B_1 and A_2B_2 meet in M.
e. A circle with center on PQ intersects PQ in A_1 and A_2 and S in B_1 and B_2 (A_1 is inside S, while B_1 is above PQ.) Prove that, if the two circles meet at 90°, then both A_1B_1 and A_2B_2 pass through M.

__Solution__. Consider the inversion in a circle with the center at the midpoint M of the arc PMQ complementary to the segment PSQ and passing through the points P and Q (see figure). Under such an inversion, line PQ is the image of the circle S, and vice versa. Hence, the tangency points A and B are the images of each other and therefore both belong to the line connecting either of them with the center of inversion, M. Furthermore, the intersection points, C and D, of a circle inscribed in a circular segment PSQ and the inversion circle with center M are invariant with respect of this inversion. Therefore, (i) MC and MD are tangent to the circle inscribed in the segment and (ii) all such inscribed circles are invariant with respect to this inversion. The above considerations substantiate (a) through (c). (d) and (e) are proven similarly, using the conformal property.

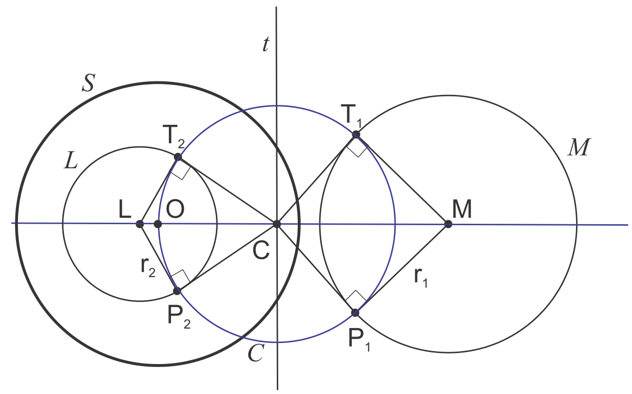
__Problem__. Steiner's Porism Theorem [Geometry Revisited, p. 124]. Given two circles - one inside the other. Pick up a point in-between and draw a circle tangent to the given two. Then draw a circle tangent to the new circle and the original two. Continue building a chain of circles each touching the two given circles and its predecessor in the chain. It may happen that, for some n, the n-th circle will touch the first circle in the chain. Prove that if this happens, it will happen regardless of the position of the starting point.



__Solution__. Consider the inversion that images the two nested circles in the problem onto the two concentric circles. Under this inversion, all nested tangent circles are imaged onto congruent circles tangent to the two concentric nested circles. The centers of these circles can be moved at will, and so can be the centers of their pre-images in the figure.

Mapping circles on concentric circles.

Theorem. For any pair of non-intersecting circles L and M (or a circle and a straight line), there exists an inversion that maps these circles onto a pair of concentric circles, L' and M'.



Proof. Consider point C of intersection of line LM connecting the centers of the two given circles and the radical axis of these circles, line t. By definition, all points on the radical axis have the same power with respect to given circles, L and M, and therefore the tangent segments from this point to both circles are equal, $|CT_1| = |CT_2| = |CP_1| = |CP_2| = R$. Here, T_1, P_1 and T_2, P_2 are tangency points on the first and the second circle, respectively (see figure). Circle with the center C and radius R passing through all four tangency points is perpendicular to both given circles and intersects line LM at point O. Consider inversion with respect to circle S with the center at this point O. Line LM will transform into itself, while circle C will transform into another line, because both pass through the center of inversion, while the two given circles will transform into another circles, L' and M'. By conformal property of inversion, the images L' and M' will be perpendicular to two intersecting straight lines, the images of line LM and circle C, because both line LM and circle C are perpendicular to given circles L and M. It then follows that circles L' and M' are concentric.

Ptolemy's inequality theorem by inversion.

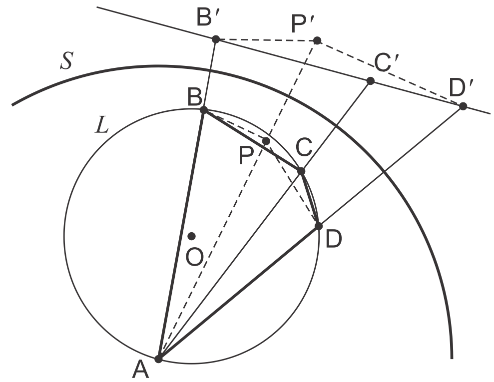
Ptolemy's inequality is an extension of Ptolemy's theorem for an inscribed quadrilateral.

Theorem. For any quadrilateral $ABCD$,

$$|AB| \cdot |CD| + |BC| \cdot |AD| \geq |AC| \cdot |BD|$$

where the equality is achieved when quadrilateral $ABCD$ is inscribed in a circle.

<u>Proof</u>. Let points A, B, C, and D be concyclic, i. e. quadrilateral $ABCD$ inscribed in a circle, L. Consider inversion with the center at the vertex of the quadrilateral, A, and radius R. It transforms cirlcle L into a line and the images of the three other vertices, points B', C', and D', lie on that line (see figure). It then follows that



$$|B'C'| + |C'D'| = |B'D'|$$

If point C is not on the circle, its image, C', is not on the line (cf points P and P' in the figure; it does not matter whether C is inside or outside the circle). Then, by triangle inequality,

$$|B'C'| + |C'D'| \geq |B'D'|$$

Using the distance formula this can be rewritten as,

$$|BC|\frac{R^2}{|AB||AC|} + |CD|\frac{R^2}{|AC||AD|} \geq |BD|\frac{R^2}{|AB||AD|}$$

Or,

$$|BC| \cdot |AD| + |CD| \cdot |AB| \geq |AC| \cdot |BD|$$