Homework for March 10, 2024.

Algebra.

Read the classwork handout. Complete the unsolved problems from the previous homework. Solve the following problems. As usual, you do not necessarily have to solve every problem. However, please solve as much as you can in the time you have. Start with those that "catch your eye", in one way or another (e.g. you think are the easiest, or most challenging). Skip the ones you already solved in the past.

1. Let x_1, x_2 and x_3 be distinct real numbers. Prove that there exists a unique polynomial, $P(x)$, of degree 2 such that $P(x_1) = 1, P(x_2) = P(x_3) = 0$. [Hint: if $P(x_1) = 0$, then $P(x)$ is divisible by $(x - x_1)$.] Find this polynomial if $x_1 = 2, x_2 = -1, x_3 = 5$.
2. As before, let x_1, x_2 and x_3 be distinct real numbers, and let y_1, y_2 and y_3 be any collection of numbers. Prove that there is a unique quadratic polynomial $f(x)$ such that $f(x_1) = y_1, f(x_2) = y_2, f(x_3) = y_3$. Find this polynomial if $x_1 = 2, x_2 = -1, x_3 = 5, y_1 = 3, y_2 = 6, y_3 = 18$. [Hint: look for in the form $f(x) = y_1 f_1(x) + \cdots$.]
3. Prove the following general result: given numbers $x_1, \ldots, x_n, y_1, \ldots, y_n$, such that x_i are distinct, there exists a unique polynomial $f(x)$ of degree $n - 1$ such that $f(x_i) = y_i, i = 1, \ldots, n$. (For $n = 2$, this is a statement that there is a unique line through two given points.)
4. Prove that if $P(x)$ is a polynomial with integer coefficients, then for any integer a, b, the difference $P(a) - P(b)$ is divisible by $a - b$.
5. Le t x_1 and x_2 be the roots of the polynomial, $x^2 + 7x - 3$. Find
   a. ${x_1}^2 + {x_2}^2$
   b. $\frac{1}{x_1} + \frac{1}{x_2}$
   c. $(x_1 - x_2)^2$
   d. ${x_1}^3 + {x_2}^3$
6. What is the maximum number of different integer roots that the following polynomial can have? How does the answer change if $a = 0$?

$$x^{10} + ax^9 + bx^8 + cx^7 + dx^6 + fx^5 + gx^4 + hx^3 + kx^2 + lx^2 + mx = 1024$$

Geometry/Trigonometry.

Review the trigonometry classwork handout. Complete the unsolved problems from the previous homework and classwork exercises. Additional reading on trigonometric functions is Gelfand & Saul, Trigonometry, Chapter 2 (pp. 42-54) and Chapters 6-8 (pp. 123-163), http://en.wikipedia.org/wiki/Trigonometric_functions http://en.wikipedia.org/wiki/Sine. Solve the following problems. Skip those that you have already solved.

1. Find all x for which,
   a. $\sin x \cos x = \frac{\sqrt{2}}{2}$
   b. $\sin x \cos x = \frac{1}{2}$
   c. $\sin x \cos x = \frac{\sqrt{3}}{4}$
2. Find the sum of the following series,

$$S = \cos x + \cos 3x + \cos 5x + \cos 7x + \cdots + \cos 2021x$$

(hint: multiply the sum by $2 \sin x$)

3. Calculate:
   a. $\cos 75° + \cos 15° =$
   b. $\cos\frac{\pi}{12} - \cos\frac{5\pi}{12} =$
4. Let A, B and C be angles of a triangle. Prove that

$$\tan\frac{A}{2}\tan\frac{B}{2} + \tan\frac{B}{2}\tan\frac{C}{2} + \tan\frac{C}{2}\tan\frac{A}{2} = 1$$

5. Prove the following equalities:
   a. $\frac{1}{\sin\alpha} + \frac{1}{\tan\alpha} = \cot\frac{\alpha}{2}$
   b. $\sin^2\left(\frac{7\pi}{8} - 2\alpha\right) - \sin^2\left(\frac{9\pi}{8} - 2\alpha\right) = \frac{\sin 4\alpha}{\sqrt{2}}$
   c. $(\cos\alpha - \cos\beta)^2 + (\sin\alpha - \sin\beta)^2 = 4\sin^2\frac{\alpha-\beta}{2}$
   d. $\frac{\cot^2 2\alpha - 1}{2\cot 2\alpha} - \cos 8\alpha \cot 4\alpha = \sin 8\alpha$
   e. $\sin^6\alpha + \cos^6\alpha + 3\sin^2\alpha\cos^2\alpha = 1$

f. $\frac{\sin 6\alpha+\sin 7\alpha+\sin 8\alpha+\sin 9\alpha}{\cos 6\alpha+\cos 7\alpha+\cos 8\alpha+\cos 9\alpha} = \tan\frac{15\alpha}{2}$

g. $\sin^6 \alpha + \cos^6 \alpha = \frac{5+3\cos 4\alpha}{8}$

h. $16\sin^5 \alpha - 20\sin^3 \alpha + 5\sin \alpha = \sin 5\alpha$

i. $\frac{\cos 64° \cos 4° - \cos 86° \cos 26°}{\cos 71° \cos 41° - \cos 49° \cos 19°}$

j. $\sin 20° \sin 40° \sin 60° \sin 80° = \frac{3}{16}$

k. $\frac{1}{\sin 10°} - \frac{\sqrt{3}}{\cos 10°} = 4$

6. Simplify the following expressions:

a. $\sin^2\left(\frac{\alpha}{2} + 2\beta\right) - \sin^2\left(\frac{\alpha}{2} - 2\beta\right)$

b. $2\cos^2 3\alpha + \sqrt{3}\sin 6\alpha - 1$

c. $\cos^4 2\alpha - 6\cos^2 2\alpha \sin^2 2\alpha + \sin^4 2\alpha$

d. $\sin^2(135° - 2\alpha) - \sin^2(210° - 2\alpha) - \sin^2 195° \cos(165° - 4\alpha)$

e. $\frac{\cos 2\alpha - \cos 6\alpha + \cos 10\alpha - \cos 14\alpha}{\sin 2\alpha + \sin 6\alpha + \sin 10\alpha + \sin 14\alpha}$

7. Solve the following equations and inequalities:

a. $\cos^2 \pi x + 4\sin \pi x + 4 = 0$

b. $\sin x + \sin 2x + \sin 3x = \cos x + \cos 2x + \cos 3x$

c. $\cos 3x - \sin x = \sqrt{3}(\cos x - \sin 3x)$

d. $\sin^2 x - 2\sin x \cos x = 3\cos^2 x$

e. $\sin 6x + 2 = 2\cos 4x$

f. $\cot x - \tan x = \sin x + \cos x$

g. $\sin x \geq \pi/2$

h. $\sin x \leq \cos x$