Homework for March 3, 2024.

## Algebra.

Read the classwork handout. Complete the unsolved problems from the previous homework. Solve the following problems. As usual, you do not necessarily have to solve every problem. However, please solve as much as you can in the time you have. Start with those that "catch your eye", in one way or another (e.g. you think are the easiest, or most challenging). Skip the ones you already solved in the past.

1. Perform long division of the following polynomials.
    a. $(x^5 - 2x^3 + 3x^2 - 4) \div (x^2 - x + 1)$
    b. $(x^4 - x^2 + 1) \div (x + 1)$
    c. $(x^7 + 1) \div (x^3 - x + 1)$
    d. $(6x^6 - 5x^5 + 4x^4 - 3x^3 + 2x - 1) \div (x^2 + 1)$
    e. $(x^5 - 32) \div (x + 2)$
    f. $(x^5 - 32) \div (x - 2)$
    g. $(x^6 + 64) \div (x^2 + 4)$
    h. $(x^6 + 64) \div (x^2 - 4)$
    i. $(x^{100} - 1) \div (x^2 - 1)$
2. Can you find coefficients $a, b$, such that there is no remainder upon division of a polynomial, $x^4 + ax^3 + bx^2 - 2x - 10$,
    a. by $x + 5$
    b. by $x^2 + x - 1$
3. Prove that,
    a. for odd $n$, the polynomial $x^n + 1$ is divisible by $x + 1$
    b. $2^{100} + 1$ is divisible by 17.
    c. $2^n + 1$ can only be prime if $n$ is a power of 2 [Primes of this form are called Fermat primes; there are very few of them. How many can you find?]
    d. for any natural number $n$, $8^n - 1$ is divisible by 7.
    e. for any natural number $n$, $15^n + 6$ is divisible by 7
4. Factor (i.e., write as a product of polynomials of smaller degree) the following polynomials.

    a. $1 + a + a^2 + a^3$
    b. $1 - a + a^2 - a^3 + a^4 - a^5$
    c. $a^3 + 3a^2b + 3b^2a + b^3$

d. $x^4 - 3x^2 + 2$

5. Simplify the following expressions using polynomial factorization.
   e. $\frac{x+y}{x} - \frac{x}{x-y} + \frac{y^2}{x^2-xy}$
   f. $\frac{x^6-1}{x^4+x^2+1}$
   g. $\frac{a^3-2a^2+5a+26}{a^3-5a^2+17a-13}$
6. Solve the following equations
   h. $\frac{x^2+1}{x} + \frac{x}{x^2+1} = 2.9$ (hint: substitution)
   i. $\frac{14}{20-6x-2x^2} + \frac{x^2+4x}{x^2+5x} - \frac{x+3}{x-2} + 3 = 0$ (hint: factorize square polynomials)
7. Write Vieta formulae for the reduced cubic equation, $x^3 + px + q = 0$. Let $x_1, x_2$ and $x_3$ be the roots of this equation. Find the following combination in terms of $p$ and $q$,
   j. $(x_1 + x_2 + x_3)^2$
   k. ${x_1}^2 + {x_2}^2 + {x_3}^2$
   l. $\frac{1}{x_1} + \frac{1}{x_2} + \frac{1}{x_3}$
   m. $(x_1 + x_2 + x_3)^3$
8. The three real numbers $x, y, z$, satisfy the equations

$$x + y + z = 7$$

$$\frac{1}{x} + \frac{1}{y} + \frac{1}{z} = \frac{1}{7}$$

   Prove that then, at least one of $x, y, z$ is equal to 7. [Hint: Vieta formulas]

9. Find all real roots of the following polynomial and factor it: $x^4 - x^3 + 5x^2 - x - 6$.

## Geometry.

Read the classwork handout. Complete the unsolved problems from the previous homework (some are repeated below) and classwork exercises. Additional reading on trigonometric functions is Gelfand & Saul,

Trigonometry, Chapter 2 (pp. 42-54) and Chapters 6-8 (pp. 123-163),
http://en.wikipedia.org/wiki/Trigonometric_functions
http://en.wikipedia.org/wiki/Sine. Solve the following problems.

## Problems.

1. Using the expressions for the sine and the cosine of the sum of two angles derived in class, derive expressions for (classwork exercise),
   a. $\sin 3\alpha$
   b. $\cos 3\alpha$
   c. $\tan(\alpha \pm \beta)$
   d. $\cot(\alpha \pm \beta)$
   e. $\tan(2\alpha)$
   f. $\cot(2\alpha)$
2. Derive the following expressions (classwork exercise):
   a. $\tan\alpha \pm \tan\beta = \frac{\sin(\alpha\pm\beta)}{\cos(\alpha)\cos(\beta)}$ $\quad \cot\alpha \pm \cot\beta = \pm\frac{\sin(\alpha\pm\beta)}{\sin(\alpha)\sin(\beta)}$
   b. $\sin^2\alpha = \frac{1}{2}(1-\cos 2\alpha)$ $\quad \cos^2\alpha = \frac{1}{2}(1+\cos 2\alpha)$
   c. $\sin 3\alpha = 3\sin\alpha - 4\sin^3\alpha$ $\quad \cos 3\alpha = 4\cos^3\alpha - 3\cos\alpha$
   d. $\sin^3\alpha = \frac{1}{4}(3\sin\alpha - \sin 3\alpha)$ $\quad \cos^3\alpha = \frac{1}{4}(3\cos\alpha + \cos 3\alpha)$
   e. $\sin\frac{\alpha}{2} = \sqrt{\frac{1}{2}(1-\cos\alpha)}$ $\quad \cos\frac{\alpha}{2} = \sqrt{\frac{1}{2}(1+\cos\alpha)}$
   f. $\tan\frac{\alpha}{2} = \sqrt{\frac{1-\cos\alpha}{1+\cos\alpha}} = \frac{1-\cos\alpha}{\sin\alpha} = \frac{\sin\alpha}{1+\cos\alpha}$
   g. $\cot\frac{\alpha}{2} = \sqrt{\frac{1+\cos\alpha}{1-\sin\alpha}} = \frac{1+\cos\alpha}{\sin\alpha} = \frac{\sin\alpha}{1-\cos\alpha}$
   h. $\sin 2\alpha = \frac{2\tan\alpha}{1+\tan^2\alpha}$
   i. $\cos 2\alpha = \frac{1-\tan^2\alpha}{1+\tan^2\alpha}$
   j. $\tan 2\alpha = \frac{2\tan\alpha}{1-\tan^2\alpha}$
3. Show that

   a. $\cos^2\alpha + \cos^2\left(\frac{2\pi}{3}+\alpha\right) + \cos^2\left(\frac{2\pi}{3}-\alpha\right) = \frac{3}{2}$

   b. $\sin\alpha + \sin\left(\frac{2\pi}{3}+\alpha\right) + \sin\left(\frac{4\pi}{3}+\alpha\right) = 0$

c. $\frac{\sin 3x}{\sin x} - \frac{\cos 3x}{\cos x} = 2$

4. Without using calculator, find:

   a. $\sin 75° =$

   b. $\cos 75° =$

   c. $\sin \frac{\pi}{8} =$

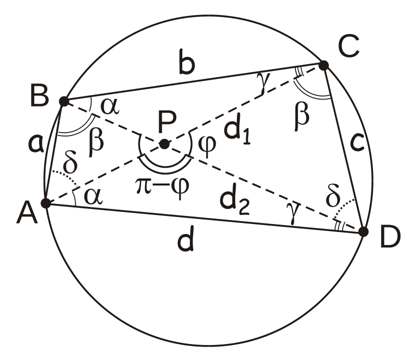
   d. $\cos \frac{\pi}{8} =$

   e. $\sin \frac{\pi}{16} =$

   f. $\cos \frac{\pi}{16} =$

5. Show that the length of a chord in a circle of unit diameter is equal to the sine of its inscribed angle.

6. Using the result of the previous problem, express the statement of the Ptolemy theorem in the trigonometric form, also known as Ptolemy identity (see Figure):




   $\sin(\alpha + \beta)\sin(\beta + \gamma) = \sin\alpha \sin\gamma + \sin\beta \sin\delta$,

   if $\alpha + \beta + \gamma + \delta = \pi$.

7. Prove the Ptolemy identity in previous problem using the addition formulas for sine and cosine.

8. Using the Sine and the Cosine theorems, prove the Heron's formula for the area of a triangle,

$$S_{\Delta ABC} = \sqrt{s(s-a)(s-b)(s-c)}$$

   where $s = \frac{a+b+c}{2}$ is the semi-perimeter.