Homework for February 4, 2024.

Algebra.

Review the classwork handout. Review the classwork and complete the exercises which were not solved in class. Try solving the unsolved problems from the previous homework (some are repeated below) and the following new problems.

1. Construct a proof that the set of all real numbers, $\mathbb{R}$, is uncountable, using the binary notation for real numbers.
2. Show that for the set of natural numbers, $\mathbb{N}$, cardinality of the set of all possible subsets is equal to that of a continuum of real numbers (Hint: use the binary number system).
3. Show that the set of points on any segment, $[a, b]$, on a line, has the same cardinality as
   a. the set of points on any other segment, $[c, d]$
   b. the set of points on a circle of unit radius
   c. the set of all points on a plane
   d. the set of all points in an n −dimensional hyper-cube
4. Show that each of the following sets has the same cardinality as a closed interval [0; 1] (i.e., there exists a bijection between each of these sets and [0; 1]).
   a. Interval [0; 1) [Hint: interval [0; 1] can be written as a union of two subsets, $A \cup B$, where A is a countable set including the interval end(s)].
   b. Open interval (0; 1)
   c. Set of all infinite sequences of 0s and 1s
   d. $\mathbb{R}$
   e. $[0,1] \times [0,1]$
5. Prove the following properties of countable sets. For any two countable sets, A, B,
   a. Union, $A \cup B$, is also countable, $\big((c(A) = \aleph_0) \wedge (c(B) = \aleph_0)\big) \Rightarrow (c(A \cup B) = \aleph_0)$
   b. Product, $A \times B = \{(a, b), a \in A, b \in B\}$, is also countable, $\big((c(A) = \aleph_0) \wedge (c(B) = \aleph_0)\big) \Rightarrow (c(A \cup B) = \aleph_0)$

c. For a collection of countable sets, $\{A_n\}, c(A_n) = \aleph_0$, the union is also countable, $c(A_1 \cup A_2 \dots \cup A_n) = \aleph_0$
d. For a collection of countable sets, $\{A_n\}, c(A_n) = \aleph_0$, the Cartesian product is also countable, $c(A_1 \times A_2 \dots \times A_n) = \aleph_0$

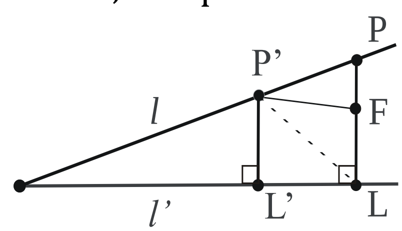
6. Write the following rational decimals in the binary system (hint: you may use the formula for an infinite geometric series).
   a. 1/15
   b. 1/14
   c. 1/7
   d. 1/6
   e. $0.33333\ldots = 0.(3)$
   f. $0.13333\ldots = 0.1(3)$

Geometry.

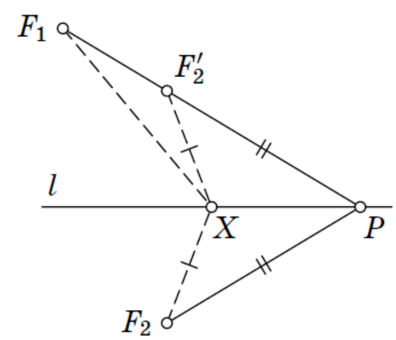
Review the previous classwork notes. Solve the problems below and the remaining problems from the previous homework.

Problems.

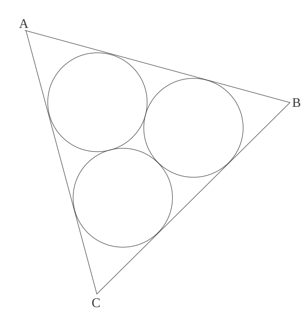
1. Given two lines, l and l', and a point F not on any of those lines, find point P on l such that the (signed) difference of distances from it to l' and F, $|P'L'| - |P'F|$, is maximal. As seen in the figure, for any P' on l the distance to l', $|P'L'| \leq |P'L| \leq |P'F| + |FL|$, where $|FL|$ is the distance from F to l'. Hence, $|P'L'| - |P'F| \leq |FL|$, and the difference is largest ($= |FL|$) when point P belongs to the perpendicular FL from point F to l'.



2. Given line l and points F_1 and F_2 lying on different sides of it, find point P on the line l such that the absolute value of the difference in distances from P to points F_1 and F_2 is maximal. As above, let $F_2{}'$ be the reflection of F_2 in l. Then for any point X on l, $|XF_2| - |XF_1'| \leq |F_1F_2'|$.



3. Find the (x, y) coordinates of the common (intersection) point of the two lines, one passing through the origin at 45 degrees to the X-axis, and the other passing through the point (1,0) at 60 degrees to it.
4. Find the (x, y) coordinates of the common (intersection) points of the parabola $y = x^2$ and of the ellipse centered at the origin and with major axis along the Y-axis whose length equals 2, and the minor axis along the X-axis whose length equals 1.



5. (Skanavi 10.122) Find the locus of the midpoints of all chords of a given circle with the center O, which intersect given chord AB of this circle.
6. Three circles of radius r touch each other. Find the area of the triangle ABC formed by tangents to pairs of circles (see figure).