Homework for December 3, 2023.

Algebra.

Review the classwork handout. Solve the remaining problems from the previous homework assignments and classwork exercises. Try solving the following problems.

Review the classwork handout and complete the exercises. Solve the remaining problems from the previous homework (you may skip the ones considered in class). Solve the following problems.

1. Using the results of the previous homework where you used the Euclidean algorithm to provide the continued fraction representation for the following numbers, construct and solve the Diophantine equations relating the numerator, denominator, and their GCD in the right-hand side.
   a. $\frac{1351}{780}$
   b. $\frac{25344}{8069}$
   c. $\frac{29376}{9347}$
   d. $\frac{6732}{1785}$
   e. $\frac{2187}{2048}$
   f. $\frac{3125}{2401}$

2. Consider the quadratic equation $x^2 = 7x + 1$. Find a continued fraction corresponding to a root of this equation.

3. Find the value of the continued fraction given by
   a. $x = \{1,1,1,1,\dots\} = 1 + \frac{1}{1+\frac{1}{1+\frac{1}{1+\cdots}}}$
   b. $x = \{1,2,2,2,\dots\} = \frac{1}{2+\frac{1}{2+\frac{1}{2+\cdots}}}$

c. $\{1,2,3,3,3,\ldots\} = 1 + \cfrac{1}{2+\cfrac{1}{3+\cfrac{1}{3+\cfrac{1}{3+\cdots}}}}$

4. Find the set of all values of x for which the following expression makes sense: $\sqrt{25 - x^2} + \frac{4}{x-2} - \frac{1}{x}$.

5. Consider the quadratic equation $x^2 = 7x + 1$. Find a continued fraction corresponding to a root of this equation.

6. Using the continued fraction representation, find rational number, r, approximating $\sqrt{2}$ to the absolute accuracy of 0.0001.

7. Consider the values of the following expression, y, for different x. How does it depend on x when n becomes larger and larger?

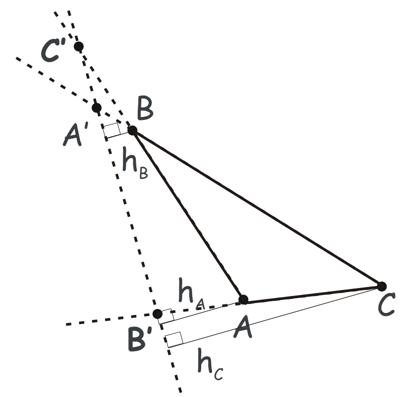
$$n\ fractions \left\{ \begin{array}{l} y = 3 - \cfrac{2}{3-\cfrac{2}{3-\cfrac{2}{3-\cdots}}} \\ \qquad\qquad\qquad \cdots - \cfrac{2}{3-x} \end{array} \right. .$$

Geometry.

Review the last classwork handout on solving problems using mass points and the center of mass. Solve the unsolved problems from previous homeworks. Try solving the following problems (skip the ones you have already solved).

Problems.

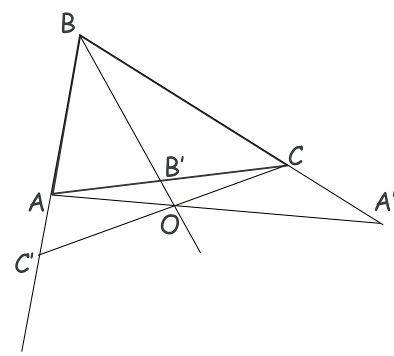
1. Prove Menelaus theorem for the configuration shown on the right using mass points. Menelaus theorem states,



Points C', A' and B', which belong to the lines containing the sides AB, BC and CA, respectively, of triangle ABC are collinear if and only if,

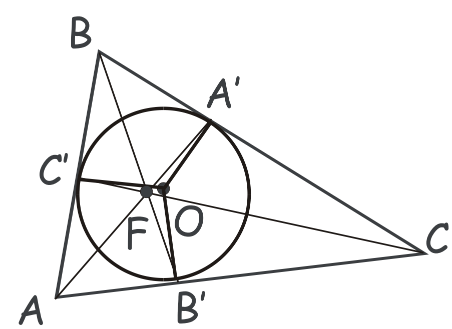
$$\frac{|AC'|}{|C'B|}\frac{|BA'|}{|A'C|}\frac{|CB'|}{|B'A|}=1$$

2. Prove the extended Ceva theorem (i) using mass points and the center of mass and (ii) using the similarity of triangles. Extended Ceva theorem states,



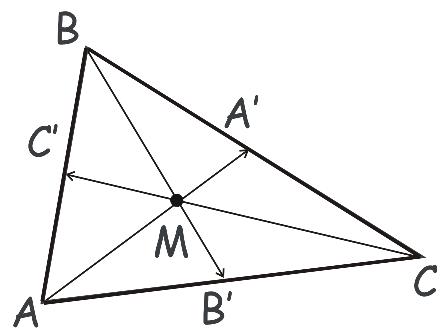
Segments (Cevians) connecting vertices A, B and C, with points A', B' and C' on the sides, or on the lines that suitably extend the sides BC, AC, and AB, of triangle ABC, are concurrent if and only if,

$$\frac{|AC'|}{|C'B|}\frac{|BA'|}{|A'C|}\frac{|CB'|}{|B'A|}=1$$



3. In a triangle ABC, A', B' and C' are the tangent points of the inscribed circle and the sides BC, AC, and AB, respectively (see Figure). Prove that cevians AA', BB' and CC' are concurrent (their common point F is called the Gergonne point).

4. Points P and Q on the lateral sides AB and BC of an isosceles triangle ABC divide these sides into segments whose lengths have ratios $|AP|:|PB| = n$, and $|BQ|:|QC| = n$. Segment PQ crosses altitude BB' at point M. What is the ratio $|BM|:|MB'|$ of two segments into which PQ divides the altitude BB'?
5. In a triangle ABC, Cevian segments AA', BB' and CC' are concurrent and cross at a point M (point C' is on the side AB, point B' is on the side AC, and point A' is on the side BC). Given the ratios $\frac{AC'}{C'B} = p$ and $\frac{AB'}{B'C} = q$, find the ratio $\frac{AM}{MA'}$ (express it through p and q).



6. Prove that in a right triangle, each side of the right angle is the geometric mean between the whole hypotenuse and its projection onto the hypotenuse. That is, if BD is the altitude from the vertex of the right angle, ABC, onto the hypotenuse, AC, then $|AB|^2 = |AC||AD|$.
7. Prove that three medians in a triangle divide it into six smaller triangles of equal area.