MATH 8 [JAN 7, 2024]
HANDOUT 14 EUCLIDEAN GEOMETRY 1: AXIOMS.

Euclidean geometry is a way describe geometric properties of various figures in the plane *exactly*. Figures are understood as sets of **points**; we will use capital letters for points and write $P \in m$ for *"point P lies in figure m"*, or *"figure m contains point P"*. A **point** is the simplest geometric object and is so basic that it can not be explained in terms of even simpler notions; everyone is believed to know what a "point" is. In addition, there are some other basic notions (**lines**, **distances**, **angle measures**) that are not be defined, and understood as elementary. Instead, we can state some basic properties of these objects; these basic properties are usually called **POSTULATES** or **AXIOMS** of Euclidean geometry. Using these basic properties, we can then apply rules of logic to deduce another property, which we call *proving a theorem*. **All results in Euclidean geometry should be proven by deducing them from the axioms**; justifications *"it is obvious"*, *"it is well-known"*, or *"it is clear from the figure"* are not acceptable! In addition to logical rules, we will also use all the usual properties of real numbers, equations, inequalities, and alike.

Note that we will make diagrams helping us understand relations between points, lines, angles, etc. Diagrams are *never perfect* (and do not have to be) because we use them only as aid to our thought process. Unlike diagrams, statements made in geometry are *exact*, and thus require more care to demonstrate than just a picture. For example, it is impossible to draw two precisely parallel lines; we can only "pretend" that their finite segments represent them in a picture, but we have to use axioms (and theorems already proven) to demonstrate additional properties. On its own, a picture is useful *only as illustration* to help understand our argument.

For your enjoyment, take a look at the book which gave rise to Euclidean geometry and much more, Euclid's *Elements*, dated about 300 BC, and used as the standard textbook for the next 2000 years. Nowadays it is available online at `http://math.clarku.edu/~djoyce/java/elements/toc.html`

1. Basic objects

These basic objects and notions are the basis of all our later constructions: we will define and discuss all other objects in terms of the basic ones. No do not need (and cannot give) definitions for these basic objects, and consider them self-evident, as well as some of their properties.

- Points (denoted by upper-case letters: A, B, ...) can be said to have zero "size" in any respect;
- Lines (denoted by lower-case letters: l, m, ...): infinite in both directions and split the plane into "half-planes";
- Distances: for any two points A, B, there is a non-negative number AB, called the distance between A, B. The distance is zero *if and only if* points exactly coincide.

Note that one can measure distances with a ruler and angles with a protractor, but only as precisely (or imprecisely) as the tool allows. In geometry, however, these measures are considered **exact numbers**.

We will also frequently use words *"between"* when describing relative position of points on a line (as in: A is between B and C) and *"inside"* (as in: point C is inside angle $\angle AOB$). We do not give full list of axioms for these notions; it is possible, but rather boring.

Having these basic notions, we can now define more objects. Namely, we can give definitions of

- interval, or line segment (notation: $\overline{AB}$): set of all points on line $\overleftrightarrow{AB}$ which are between A and B, together with points A and B themselves. The segment length (denoted as AB or $|AB|$) is the distance between A and B.
- ray, or half-line (notation: $\overrightarrow{AB}$): set of all points on the line $\overleftrightarrow{AB}$ which are on the same side of A as B (Note that we have not defined the concept *"on the same side"* but will be using it in the future).
- angle (notation: $\angle AOD$): figure consisting of two rays ($\overrightarrow{OA}$ and $\overrightarrow{OD}$) with a common vertex (O). For any angle $\angle ABC$, there is a non-negative real number $m\angle ABC$, called the measure of this angle.
- parallel lines: two distinct lines l, m are called parallel (notation: $l \parallel m$) if they do not intersect, i.e. have no common points. We also say that every line is parallel to itself (it is a rather convenient convention, which will make our lives easier – the intuition here is that parallel lines have the same "direction"). Also, the angle between two parallel rays is zero.

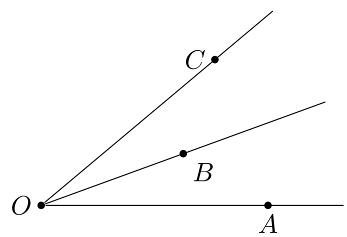
2. First axioms

After we introduced some objects, including undefined ones, we need to have statements (*axioms*) that describe their properties. Of course, the lack of definition for undefined objects makes such properties impossible to prove. The goal here is to state the *minimal number* of such properties that we take for granted, just enough to be able to prove or derive harder and more complicated statements. Here are the first few axioms:

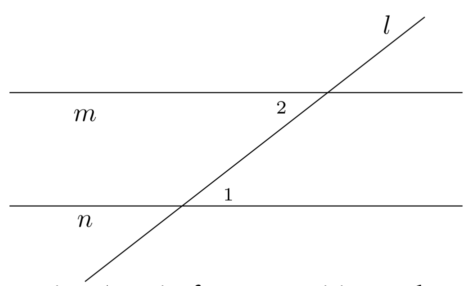
Axiom 1. *For any two distinct points A, B, there is exactly one line to which both these points belong. (This line is usually denoted $\overleftrightarrow{AB}$). In other words, two distinct points are sufficient (and necessary) to specify a line.*

Axiom 2. *If distinct points A, B, C are on the same line, exactly one is* between the other two*; if point B is between A and C, then $AC = AB + BC$.*

Axiom 3. *If point B is inside angle $\angle AOC$, then $m\angle AOC = m\angle AOB + m\angle BOC$. Also, the measure of a straight angle is equal to $180°$.*



Axiom 4. *Let line l intersect lines m, n and angles $\angle 1$, $\angle 2$ are as shown in the figure below (in this situation, such a pair of angles is called alternate interior angles). Then $m \parallel n$ if and only if $m\angle 1 = m\angle 2$.*



In addition, we will assume that given a line l and a point A on it, for any positive real number d, there are exactly two points on l at distance d from A, on opposite sides of A, and similarly for angles: given a ray and angle measure, there are exactly two angles with that measure having that ray as one of the sides.

Homework

All problems below are important — please try to finish them all! Note that you may use all results that are presented in the previous sections. This means that you may use any theorem if you find it a useful logical step in your proof. The only exception is when you are explicitly asked to prove a given theorem, in which case you must understand how to draw the result of the theorem from previous theorems and axioms. Each step of your proof must be based on some previous, already proven, statement or an axiom.

1. It is important that you know some geometry notation.
   (a) What does the symbol $\parallel$ mean? How do you pronounce it? How would you read "$a \parallel b$"?
   (b) What does the symbol $\perp$ mean? How would you say "$a \perp b$"?
   (c) Suppose you have two points X and Y. What is the difference between $\overline{XY}$, $\overleftrightarrow{XY}$, $\overrightarrow{XY}$? What are each of these things called?
   (d) Given three points E, F, G, what does $EF + FG$ mean?
   (e) Given four points A, B, C, D, what does $m\angle ADC + m\angle BDC$ mean? If I tell you $m\angle ADC + m\angle BDC = 180°$, does that tell you any information about $m\angle ADC$ or $m\angle BDC$?
   (f) What does the symbol $\triangle$ mean? For example, if A and B and C are points, what is $\triangle ABC$?

2. (a) What is a proof? Give an example. Can you come up with an example that is not about geometry?
   (b) What is an axiom? Give an example. Can you come up with an example that is not about geometry?

3. In this problem, you will make diagrams. Part of the purpose of this exercise is so that, when you think about geometry, the pictures in your notes or in your mind aren't all just the diagrams I draw out for you in class or on classwork sheets. You have to be able to draw or visualize configurations of lines other than the way they're set up in axiom 4, for example.

(a) Given lines a, b, c, is it possible that $a \parallel b$ and $\neg(b \parallel c)$ but $a \parallel c$? Draw a diagram and then explain your reasoning on how to answer this question. ("explain" means, of course, in writing.)

(b) Suppose we have parallel lines l, m. Let A, B, C be points on l, with B between A, C. Let X, Y, Z be points on m, with Y between X, Z. Is it possible for lines $\overleftrightarrow{AX}$, $\overleftrightarrow{BY}$, $\overleftrightarrow{CZ}$ to all intersect at one point? Draw a diagram of what this might look like.

(c) Consider the diagram you drew in the previous part, with the lines l, m and the six points, and the three cross-lines that intersect at a point. Now consider the lines $\overleftrightarrow{AZ}$, $\overleftrightarrow{CX}$. Do these two lines intersect at a point on $\overleftrightarrow{BY}$? Draw a diagram where this *is* the case, and then draw a second diagram where this *is not* the case.

(d) Draw a rectangle that's not a square, and draw it so that one of the bases is horizontal. Then draw one of the rectangle's diagonals. Notice that, of the two right angles formed at the rectangle's base, the rectangle's diagonal splits one of those angles into two smaller angles. Which of the two angles is bigger - the one below the diagonal, or the one above the diagonal? Draw a second rectangle where the opposite relation holds true (for example, if the lower angle was bigger in your first rectangle, draw a second rectangle where the lower angle split by the diagonal is smaller).

4. Can you formulate Axiom 4 without referring to the picture (i.e. without using any statement such as "angles $\angle 1$, $\angle 2$ are as shown in the figure below"? You will have to introduce a number of points and have very clear notations.

5. The following logic and geometric statements come in equivalent pairs. Each logic statement has exactly one geometric statement that is equivalent to it. Match these statements into their equivalent pairs, with an explanation of why the pairs you chose are equivalent. [**Note**: the quantifier $\exists!$ stands for "there exists a unique…", and $\emptyset$ is an empty set.]

Geometric statements:

(a) For any two distinct points there is a unique line containing these points.

(b) Given a line and a point not on the line there exists a unique line though the given point that is parallel to the given line.

(c) If two lines are parallel and another line intersects one of them, then it intersects the other one as well.

(d) If two lines are parallel to the same line, then they are parallel to each other

Logic statements:

(a) $\forall l\ \forall m$ such that $l \parallel m$ $[\forall n\ (n \cap l \neq \emptyset \rightarrow n \cap m \neq \emptyset)]$

(b) $\forall A\ \forall B$ such that $A \neq B$ $\quad [\exists!\ l\ (A \in l \wedge B \in l)]$

(c) $\forall l\ \forall m$ $[(\exists n$ such that $n \parallel l \wedge n \parallel m) \rightarrow (l \parallel m)]$

(d) $\forall l\ \forall A$ such that $A \notin l$ $\quad [\exists!\ m\ (A \in m \wedge m \parallel l)]$