

MATH 8: HANDOUT 19 [FEB 11, 2024]
EUCLIDEAN GEOMETRY 6: CIRCLES

CIRCLES

Definition. A circle with center O and radius $r > 0$ is the set of all points P in the plane such that $OP = r$.

Traditionally, one denotes circles by Greek letters: λ, ω, \dots

Given a circle λ with center O ,

- A radius is any line segment from O to a point A on λ ,
- A chord is any line segment between distinct points A, B on λ ,
- A diameter is a chord that passes through O ,
- A line is tangent if it intersects the circle at one point, and is said to be the tangent through that point.
- Two circles are tangent if they intersect at exactly one point.

Theorem 21. Let A be a point on circle λ centered at O , and m a line through A . Then m is tangent to λ if and only if $m \perp \overline{OA}$. Moreover, there is exactly one tangent to λ at A .

Proof. First we prove $(m \text{ is tangent to } \lambda) \implies (m \perp \overline{OA})$. Suppose m is tangent to λ at A but not perpendicular to \overline{OA} . Let \overline{OB} be the perpendicular to m through O , with B on m . Construct point C on m such that $BA = BC$; then we have that $\triangle OBA \cong \triangle OBC$ by *SAS*, using $OB = OB$, $\angle OBA = \angle OBC = 90^\circ$, and $BA = BC$. Therefore $OC = OA$ and hence C is on λ . But this means that m intersects λ at two points, which is a contradiction.

Now we prove $(m \perp \overline{OA}) \implies (m \text{ is tangent to } \lambda)$. Suppose m passes through A on λ such that $m \perp \overline{OA}$. If m also passed through B on λ , then $\triangle AOB$ would be an isosceles triangle since $\overline{AO}, \overline{BO}$ are radii of λ . Therefore $\angle ABO = \angle BAO = 90^\circ$, i.e. $\triangle AOB$ is a triangle with two right angles, which is a contradiction. □

Notice that, given point O and line m , the perpendicular \overline{OA} from O to m (with A on m) is the shortest distance from O to m , therefore the locus of points of distance exactly OA from O should line entirely on one side of m . This is essentially the idea of the above proof.

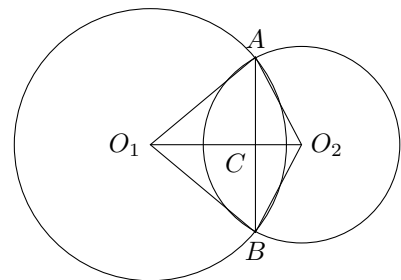
Theorem 22. Let \overline{AB} be a chord of circle λ with center O . Then O lies on the perpendicular bisector of \overline{AB} . Moreover, if C is on \overline{AB} , then C bisects \overline{AB} if and only if $\overline{OC} \perp \overline{AB}$.

Proof. Let m be the perpendicular bisector of \overline{AB} . The center O of λ is equidistant from A, B by the definition of a circle, therefore O must be on m . Let m intersect \overline{AB} at D . We then have that D is the midpoint of \overline{AB} and also the foot of the perpendicular from O to \overline{AB} .

Then if C bisects \overline{AB} , C lies on the perpendicular bisector m of \overline{AB} , which passes through O , thus $\overline{OC} \perp \overline{AB}$. Lastly if $\overline{OC} \perp \overline{AB}$, then because there is only one perpendicular to \overline{AB} through O , we must have $C = D$ and hence C is the midpoint of \overline{AB} . □

Theorem 23. Let ω_1, ω_2 be circles with centers at points O_1, O_2 that intersect at points A, B . Then $\overline{AB} \perp \overline{O_1O_2}$.

Proof. Let l be the perpendicular bisector of AB . By the previous theorem, l contains both centers: $O_1 \in l, O_2 \in l$. Thus, $l = \overline{O_1O_2}$, so $\overline{O_1O_2}$ is the perpendicular bisector of AB ; in particular, they are perpendicular. □



Theorem 24. (Relative positions of lines and circles) Let λ be a circle of radius r with center at O and let l be a line. Let d be the distance from O to l , i.e. the length of the perpendicular OP from O to l . Then:

- If $d > r$, then λ and l do not intersect.
- If $d = r$, then λ intersects l at exactly one point P , the base of the perpendicular from O to l . In this case, we say that l is tangent to λ at P .
- If $d < r$, then λ intersects l at two distinct points.

Proof. First two parts easily follow from the fact that a perpendicular is the (shortest) distance from a point to a line. In the last part, it is easy to show that λ can not intersect l at more than 2 points. Proving that it does intersect l precisely at two points is very hard and requires deep results about real numbers. This proof will not be given here. \square

Note that it follows from the definition that a tangent line is perpendicular to the radius OP at point of tangency. Converse is also true.

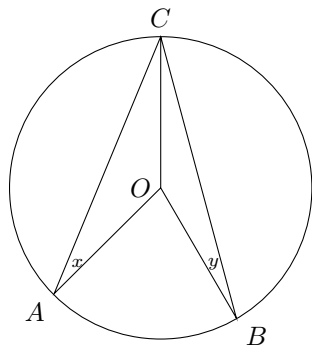
Theorem 25. Let ω_1, ω_2 be circles that are both tangent to line m at point A . Then these two circles have only one common point, A . Such circles are called tangent.

Proof. By Theorem 21, radiuses O_1A and O_2A are both perpendicular to m at A ; since there can only be one perpendicular line to m at given point, it means that O_1, O_2 , and A are on the same line, and that m is perpendicular to O_1O_2 at A .

Now, suppose that ω_1, ω_2 intersect at point $B \neq A$. Then by the previous theorem, $\overline{AB} \perp \overline{O_1O_2}$, therefore both \overline{AB} and m are perpendicular to $\overline{O_1O_2}$ through A . We must therefore have that B is on m , but m is tangent to ω_1 through A , thus has only one intersection with ω_1 , which is a contradiction. \square

ARCS AND ANGLES

Consider a circle λ with center O , and an angle formed by two rays from O . Then these two rays intersect the circle at points A, B , and the portion of the circle inside this angle is called the arc subtended by $\angle AOB$.



Theorem 26. Let A, B, C be on circle λ with center O . Then $\angle ACB = \frac{1}{2}\angle AOB$. The angle $\angle ACB$ is said to be inscribed in λ .

Proof. There are actually a few cases to consider here, since C may be positioned such that O is inside, outside, or on the angle $\angle ACB$. We will prove the first case here, which is pictured on the left.

Case 1. Draw in segment \overline{OC} . Denote $m\angle A = x, m\angle B = y$. Since $\triangle AOC$ is isosceles, $m\angle AC O = x$; similarly $m\angle BCO = y$, so $m\angle ACB = x + y$, and $m\angle AOC = 180^\circ - 2x, m\angle BOC = 180^\circ - 2y$. Therefore, $m\angle AOC + m\angle BOC = 360^\circ - 2(x + y)$. This implies $m\angle AOB = 2(x + y)$. \square

As a result of Theorem 26, we get that any triangle $\triangle ABC$ on λ where \overline{AB} is a diameter must be a right triangle, since the angle $\angle ACB$ has half the measure of angle $\angle AOB$, which is 180° .

The idea captured by the concept of an arc and Theorem 26 is that there is a fundamental relationship between angles and arcs of circles, and that the angle 360° can be thought of as a full circle around a point.

HOMEWORK

1. Prove that, given a segment \overline{AB} , there is a unique circle with diameter \overline{AB} .
2. Given lines $\overleftrightarrow{AB} \parallel \overleftrightarrow{CD}$ such that $\overline{AD}, \overline{BC}$ intersect at E and $AE = ED$, prove that $BE = EC$.
3. Prove that if a diameter of circle λ is a radius of circle ω , then λ, ω are tangent.
4. Complete the proof of Theorem 26 by proving the cases where O is not inside the angle $\angle ACB$. [Hint: for one of the cases, you may need to write $\angle ACB$ as the difference of two angles.]
5. Prove the converse of Theorem 26: namely, if λ is a circle centered at O and A, B , are on λ , and there is a point C such that $m\angle ACB = \frac{1}{2}m\angle AOB$, then C lies on λ . [Hint: we need to prove that $OC = OA$; consider using a proof by contradiction]
6. Let A, B be on circle λ centered at O and m the tangent to λ at A . Let C be on m such that C is on the same side of \overleftrightarrow{OA} as B . Prove that $m\angle BAC = \frac{1}{2}m\angle BOC$. [Hint: extend \overline{OA} to intersect λ at point D so that \overline{AD} is a diameter of λ . What arc does $\angle DAB$ subtend?]
7. Prove that, given two distinct points A, B on circle λ which are on the same side of diameter \overline{CD} of λ , that $\overline{CB} \neq \overline{CA}$.
8. Let $\overline{AB}, \overline{CD}$ both have midpoint E and let F, G be points such that $BECF$ and $AEDG$ are parallelograms. Prove that E is the midpoint of \overline{FG} .