**MATH 8: HANDOUT 17 [JAN 28, 2024]**
**EUCLIDEAN GEOMETRY 4: QUADRILATERALS. MIDLINE OF A TRIANGLE.**

## 9. SPECIAL QUADRILATERALS

In general, a figure with four sides (and four enclosed angles) is called a quadrilateral; by convention, their vertices are labeled in order going around the perimeter (so, for example, in quadrilateral $ABCD$, vertex $A$ is opposite vertex $C$). In case it is unclear, we use 'opposite' to refer to pieces of the quadrilateral that are on opposite sides, so side $\overline{AB}$ is opposite side $\overline{CD}$, vertex $A$ is opposite vertex $C$, angle $\angle A$ is opposite angle $\angle C$ etc.

Among all quadrilaterals, there are some that have special properties. In this section, we discuss three such types.

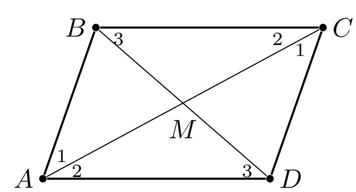
**Definition.** A quadrilateral is called

- a parallelogram, if both pairs of opposite sides are parallel
- a rhombus, if all four sides have the same length
- a trapezoid, if one pair of opposite sides are parallel (these sides are called bases) and the other pair is not.

These quadrilaterals have a number of useful properties.

**Theorem 14.** *Let $ABCD$ be a parallelogram. Then*

- $AB = DC$, $AD = BC$
- $m\angle A = m\angle C$, $m\angle B = m\angle D$
- *The intersection point $M$ of diagonals $AC$ and $BD$ bisects each of them.*




*Proof.* Consider triangles $\triangle ABC$ and $\triangle CDA$ (pay attention to the order of vertices!). By Axiom 4 (alternate interior angles), angles $\angle CAB$ and $\angle ACD$ are equal (they are marked by 1 in the figure); similarly, angles $\angle BCA$ and $\angle DAC$ are equal (they are marked by 2 in the figure). Thus, by ASA, $\triangle ABC \cong \triangle CDA$. Therefore, $AB = DC$, $AD = BC$, and $m\angle B = m\angle D$. Similarly one proves that $m\angle A = m\angle C$.

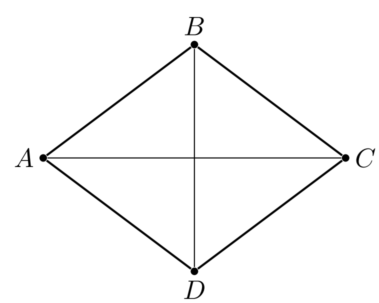
Now let us consider triangles $\triangle AMD$ and $\triangle CMB$. In these triangles, angles labeled 2 are congruent (discussed above), and by Axiom 4, angles marked by 3 are also congruent; finally, $AD = BC$ by previous part. Therefore, $\triangle AMD \cong \triangle CMB$ by ASA, so $AM = MC$, $BM = MD$. □

There are several ways you can recognize a parallelogram; in fact, conclusions in Theorem 14 are not only necessary, but also sufficient for a quadrilateral to be a paralellogram. Let's remember them as a single theorem:

**Theorem 15.** *Any quadrilateral $ABCD$ is a parallelogram if either of the following is true:*

- *its opposite sides are equal ($AB = CD$ and $AD = BC$),* ***OR***
- *two opposite sides are equal and parallel ($AB = CD$ and $AB \parallel CD$),* ***OR***
- *its diagonals bisect each other ($AM = CM$ and $BM = DM$, where $AC \cap BD = M$),* ***OR***
- *its opposing angles are equal ($\angle BAD = \angle BCD$ and $\angle ABC = \angle ADC$).*

Proofs are left to you as a homework exercise.

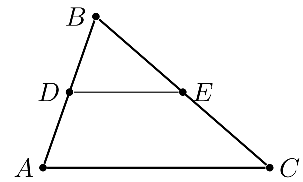



**Theorem 16.** *Let $ABCD$ be a rhombus. Then it is a parallelogram; in particular, the intersection point of diagonals is the midpoint for each of them. Moreover, the diagonals are perpendicular.*

*Proof.* Since the opposite sides of a rhombus are equal, it follows from Theorem 15 that the rhombus is a parallelogram, and thus the diagonals bisect each other. Let $M$ be the intersection point of the diagonals; since triangle $\triangle ABC$ is isosceles, and $BM$ is a median, by Theorem 12, it is also the altitude. □

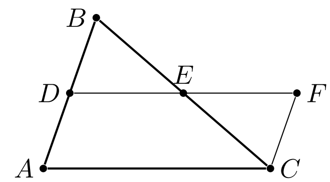
## 10. Midline of a triangle

Properties of parallelograms are very useful for proving theorems, for example about a triangle midline.




**Definition.** A midline of a triangle $\triangle ABC$ is the segment connecting midpoints of two sides.

**Theorem 17.** *If $DE$ is the midline of $\triangle ABC$, then $DE = \frac{1}{2}AC$, and $\overline{DE} \parallel \overline{AC}$.*




*Proof.* Continue line $DE$ and mark on it point $F$ such that $DE = EF$.

1. $\triangle DEB \cong \triangle FEC$ by SAS: $DE = EF$, $BE = EC$, $\angle BED \cong \angle CEF$.
2. $ADFC$ is a parallelogram: First, we can see that since $\triangle DEB \cong \triangle FEC$, then $\angle BDE \cong \angle CFE$, and since they are alternate interior angles, $AD \parallel FC$. Also, from the same congruency, $FC = BD$, but $BD = AD$ since $D$ is a midpoint. Then, $FC = DA$. So we have $FC = DA$ and $FC \parallel DA$, and therefore $ADFC$ is a parallelogram.
3. That gives us the second part of the theorem: $DE \parallel AC$. Also, since $ADFC$ is a parallelogram, $AC = DF = 2 \cdot DE$, and from here we get $DE = \frac{1}{2}AC$.

□

Alternatively, one can prove that a line parallel to one side of the triangle crosses another side in the middle, then it is a midline, and will cross the third side also in the middle.

### Homework

**Note that you may use all results that are presented in the previous sections.** This means that you may use any theorem if you find it a useful logical step in your proof. The only exception is when you are explicitly asked to prove a given theorem, in which case you must understand how to draw the result of the theorem from previous theorems and axioms.

1. Prove that in a parallelogram, sum of two adjacent angles is equal to $180°$:
$$m\angle A + m\angle B = m\angle B + m\angle C = \cdots = 180°$$
2. **[We may have done some in class - you still need to write the proofs neatly]** Prove Theorem 15: that a quadrilateral is a parallelogram if
   (a) it has two pairs of equal sides;
   (b) if two of its sides are equal and paraallel;
   (c) if its diagonals bisect each other;
   (d) if its opposite angles are equal.
   *Any of the above statements can be used as the definition of a parallelogram.*
3. (Rectangle) A quadrilateral is called rectangle if all angles have measure $90°$.
   (a) Show that each rectangle is a parallelogram.
   (b) Show that opposite sides of a rectangle are congruent.
   (c) Prove that the diagonals of a rectangle are congruent.
   (d) Prove that conversely, if $ABCD$ is a parallelogram such that $AC = BD$, then it is a rectangle.
4. Prove that in any triangle, the three perpendicular side bisectors intersect at a single point (compare with the similar fact about perpendicular bisectors – Problem 3 from Handout 16)
5. Show that in any triangle, its three midlines divide the original triangle into four triangles, all congruent to each other.
6. $*$ Prove that in any triangle, its altitudes intersect at the same point.
   *Hint: consider a triangle and its three midlines from the previous problem; draw the perpendicular side bisectors to each side of the big triangle. Are they altitudes in some other triangle?*