MATH 8 HANDOUT 15 [JAN 14, 2024]
EUCLIDEAN GEOMETRY 2: FIRST THEOREMS. PARALLEL LINES. TRIANGLES.

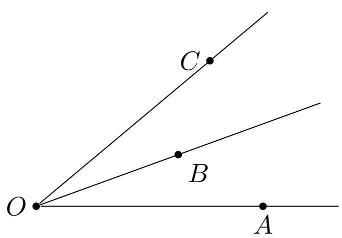
1. First axioms

After we introduced some objects, including undefined ones, we need to have statements (*axioms*) that describe their properties. Of course, the lack of definition for undefined objects makes such properties impossible to prove. The goal here is to state the *minimal number* of such properties that we take for granted, just enough to be able to prove or derive harder and more complicated statements. Here are the first few axioms:

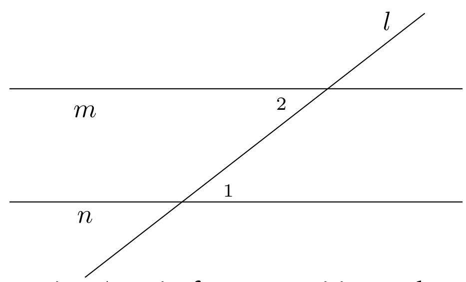
Axiom 1. *For any two distinct points A, B, there is a unique line containing these points (this line is usually denoted $\overleftrightarrow{AB}$).*

Axiom 2. *If points A, B, C are on the same line, and B is between A and C, then $AC = AB + BC$*

Axiom 3. *If point B is inside angle $\angle AOC$, then $m\angle AOC = m\angle AOB + m\angle BOC$. Also, the measure of a straight angle is equal to $180°$.*



Axiom 4. *Let line l intersect lines m, n and angles $\angle 1$, $\angle 2$ are as shown in the figure below (in this situation, such a pair of angles is called alternate interior angles). Then $m \parallel n$ if and only if $m\angle 1 = m\angle 2$.*



In addition, we will assume that given a line l and a point A on it, for any positive real number d, there are exactly two points on l at distance d from A, on opposite sides of A, and similarly for angles: given a ray and angle measure, there are exactly two angles with that measure having that ray as one of the sides.

2. First theorems

Now we can proceed with proving some results based on the axioms above.

Theorem 1. *If distinct lines l, m intersect, then they intersect at exactly one point.*

Proof. Proof by contradiction: Assume that they intersect at more than one point. Let P, Q be two of the points where they intersect. Then both l, m go through P, Q. This contradicts Axiom 1. Thus, our assumption (that l, m intersect at more than one point) must be false. □

Theorem 2. *Given a line l and point P not on l, there exists a unique line m through P which is parallel to l.*

Proof. Here we have to prove two things: the existence of a parallel line through the given point not on the given line, and its uniqueness. Below we provide a sketch of the proof – please fill in the details and draw a diagram at home!

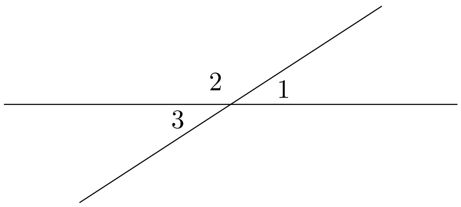
Existence: Let m be any line that goes through P and intersect l at point O. Let A be a point on the line l. Then we can measure the angle $\angle POA$. Now, let PB be such that $m\angle BPO = m\angle POA$ and B is on the other side of m than A. In this case, by Axiom 4, $\overleftrightarrow{PB} \parallel l$.

Uniqueness: Imagine that there are two lines m, n that are parallel to l and go through P. Take a line k that goes through P and intersects l in point O. Let A be a point on line l distinct from O, and B, C — points on lines m and n respectively on the other side of line k than A. Since both m, n are parallel to l, we can see that $m\angle AOP = m\angle BPO = m\angle CPO$ – but that would mean that lines $\overleftrightarrow{BP}$ and $\overleftrightarrow{CP}$ are the same — contradiction to our assumption that there are two such lines. □

Theorem 3. *If $l \parallel m$ and $m \parallel n$, then $l \parallel n$*

Proof. Assume that l and n are not parallel and intersect at point P. But then it appears that there are two lines that are parallel to m are go through point P — contradiction with Theorem 2. □

Theorem 4. *Let A be the intersection point of lines l, m, and let angles $1, 3$ be as shown in the figure below (such a pair of angles are called vertical). Then $m\angle 1 = m\angle 3$.*



Proof. Let angle 2 be as shown in the figure to the left. Then, by Axiom 3, $m\angle 1 + m\angle 2 = 180°$, so $m\angle 1 = 180° - m\angle 2$. Similarly, $m\angle 3 = 180° - m\angle 2$. Thus, $m\angle 1 = m\angle 3$. □

Theorem 5. *Let l, m be intersecting lines such that one of the four angles formed by their intersection is equal to $90°$. Then the three other angles are also equal to $90°$. (In this case, we say that lines l, m are perpendicular and write $l \perp m$.)*

Proof. Left as a homework exercise. □

Theorem 6. *Let l_1, l_2 be perpendicular to m. Then $l_1 \parallel l_2$.*
Conversely, if $l_1 \perp m$ and $l_2 \parallel l_1$, then $l_2 \perp m$.

Proof. Left as a homework exercise. □

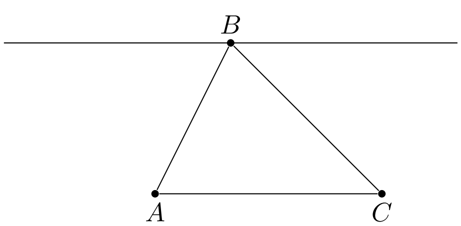
Theorem 7. *Given a line l and a point P not on l, there exists a unique line m through P which is perpendicular to l.*

Proof. Left as a homework exercise. □

3. Triangles

Theorem 8. *Given any three points A, B, C, which are not on the same line, and line segments $\overline{AB}$, $\overline{BC}$, and $\overline{CA}$, we have $m\angle ABC + m\angle BCA + m\angle CAB = 180°$. (Such a figure of three points and their respective line segments is called a triangle, written $\triangle ABC$. The three respective angles are called the triangle's interior angles.)*

Proof. The proof is based on the figure below and use of Alternate Interior Angles axiom. Details are left to you as a homework.



□

4. Congruence

It will be helpful, in general, to have a way of comparing geometric objects to tell whether they are the same. We will build up such a notion and call it congruence of objects. To begin, we define congruence of angles and congruence of line segments (note that an angle cannot be congruent to a line segment; the objects have to be the same type).

- If two angles $\angle ABC$ and $\angle DEF$ have equal measure, then they are congruent angles, written $\angle ABC \cong \angle DEF$.
- If the distance between points A, B is the same as the distance between points C, D, then the line segments $\overline{AB}$ and $\overline{CD}$ are congruent line segments, written $\overline{AB} \cong \overline{CD}$.

- If two triangles $\triangle ABC$, $\triangle DEF$ have respective sides and angles congruent, then they are congruent triangles, written $\triangle ABC \cong \triangle DEF$. In particular, this means $\overline{AB} \cong \overline{DE}$, $\overline{BC} \cong \overline{EF}$, $\overline{CA} \cong \overline{FD}$, $\angle ABC \cong \angle DEF$, $\angle BCA \cong \angle EFD$, and $\angle CAB \cong \angle FDE$.

Note that congruence of triangles is sensitive to which vertices on one triangle correspond to which vertices on the other. Thus, $\triangle ABC \cong \triangle DEF \implies \overline{AB} \cong \overline{DE}$, and it can happen that $\triangle ABC \cong \triangle DEF$ but $\neg(\triangle ABC \cong \triangle EFD)$.

5. Congruence of Triangles

Triangles consist of six pieces (three line segments and three angles), but some notion of constancy of shape in triangles is important in our geometry. We describe below some rules that allow us to, in essence, uniquely determine the shape of a triangle by looking at a specific subset of its pieces.

Axiom 5 (SAS Congruence)**.** *If triangles $\triangle ABC$ and $\triangle DEF$ have two congruent sides and a congruent included angle (meaning the angle between the sides in question), then the triangles are congruent. In particular, if $\overline{AB} \cong \overline{DE}$, $\overline{BC} \cong \overline{EF}$, and $\angle ABC \cong \angle DEF$, then $\triangle ABC \cong \triangle DEF$.*

Other congruence rules about triangles follow from the above: the ASA and SSS rules. However, their proofs are less interesting than other problems about triangles, so we can take them as axioms and continue.

Axiom 6 (ASA Congruence)**.** *If two triangles have two congruent angles and a corresponding included side, then the triangles are congruent.*

Axiom 7 (SSS Congruence)**.** *If two triangles have three sides congruent, then the triangles are congruent.*

Homework

1. (Parallel and Perpendicular Lines) Part of the spirit of Euclidean geometry is that parallelism and perpendicularity are special concepts; Theorem 6, for example, is generally considered part of the heart of Euclidean geometry. For this problem, prove the following theorems presented in the First Theorems section, using only the information from the Basic Objects and First Postulates sections. Axiom 4 will be of key importance.
   (a) Study the proof of Theorem 2 and draw a diagram that illustrates it.
   (b) Study the proof of Theorem 3.
   (c) Prove Theorem 5.
   (d) Prove Theorem 6.
   (e) Prove Theorem 7.
2. Complete the proof of Theorem 8, about sum of angles of a triangle.
3. What is the sum of angles of a quadrilateral? of a pentagon?
4. Notice that SSA and AAA are not listed as congruence rules.
   (a) Describe a pair of triangles that have two congruent sides and one congruent angle but are not congruent triangles.
   (b) Describe a pair of triangles that have three congruent angles but are not congruent triangles.
5. Prove that the following two properties of a triangle are equivalent:
   (a) All sides have the same length.
   (b) All angles are $60°$.
   A triangle satisfying these properties is called *equilateral*.
6. A triangle in which two sides are congruent is called *isosceles*. Such triangles have many special properties.
   (a) Let $\triangle ABC$ be an isosceles triangle, with $\overline{AB} \cong \overline{BC}$. Suppose D is a point on $\overline{AC}$ such that $\overline{AD} \cong \overline{DC}$ (such point is called *midpoint* of the segment). Prove that then, $\triangle ABD \cong \triangle CBD$ and deduce from this that $\angle DBA \cong \angle DBC$, and $\angle A \cong \angle C$. What can we say about $\angle ADB$?
   (b) Conversely, show that if $\triangle ABC$ is such that $\angle A \cong \angle C$, then $\triangle ABC$ is isosceles, with $\overline{AB} \cong \overline{BC}$.