Handout 16: Polynomial inequalities

1. Solving polynomial inequalities using the interval method

So far, we have solved quadratic and rational inequalities using linear inequalities. We can also consider polynomial inequalities: they would have terms like $x^2; x^3$,etc. The general rule for solving polynomial inequalities is as follows:

- Find the roots and factor your polynomial, writing it in the form $p(x) = a(x - x_1)(x - x_2)$ (for polynomial of degree more than 2, you would have more factors).
- With the roots $x_1; x_2; \dots$: divide the real line into intervals; starting with the first interval, choose a number from that interval to be your x, substitute it in the factored inequality, and determine the sign of each factor in your inequality. Then determine the sign of the product of all factors. Repeat for each interval.
- If the inequality has ≥ or ≤ signs, you should also include the roots themselves into the intervals.
- The intervals whose signs match the sign of the inequality are your solutions.

Example 1. $x^2 + x - 2 > 0$

Solution. We find the roots of the equation $x^2 + x + 2 = 0$ to be $x = -2;\ 1$. The inequality in factored form becomes $(x + 2)(x - 1) > 0$, and the roots $-2, 1$ divide the real line into three intervals $(-\infty\ ; -2); (-2; 1); (1; +\infty)$.
It is easy to see that the polynomial $x^2 + x + 2$ is positive on the first and the third intervals and negative on the second one. The solution of the inequality is then all x in interval one and three ($x < -2\ OR\ x > 1$). We write this also as $x \in (-\infty; -2) \cup (1; +\infty)$. (sign ∪ means "or"). Solving polynomial inequalities of second and higher order makes us realize that if we determine the sign of the first interval, the signs of the following intervals alternate. The graph of the polynomial crosses the $x - axis$ from above ("$+$" interval), goes below ("$-$" interval), … the curve "snakes" around the axis when crossing the roots. This is why this method for solving polynomial inequalities is also known as "snake" method. ***Careful*** – if a factor is raised at an even power, the sign will always be positive and the alternation will not apply.

2. Using the snake method

This is the interval method where you only determine the sign of the inequality in the first interval. Then you "snake" around the roots. Careful! If the root is included in a factor raised at an even power, for example the root 3 is in $(x - 3)^6$, the sign will not change.

Example 2. Solve the inequality $(x + 1)(x - 2)^2(x - 4)^3 \geq 0$.

Solution. Notice that if we solve the corresponding equation $(x + 1)(x - 2)^2(x - 4)^3 = 0$, we get $x = -1;\ 2;\ 4$.
Therefore, we need to consider the following 4 intervals:$(-\infty; -1); (-1; 2); (2; 4); (4; +\infty)$.
Notice that in the 1st interval, the expression $(x + 1)(x - 2)^2(x - 4)^3$ is positive, and therefore satisfies the inequality. Then, as x "crosses" point 1, the expression changes its sign to '$-$', and therefore the interval $(-1; 2)$ does not satisfy the inequality. Now, crossing point 2 again won't change the sign of the expression, because $(x - 2)^2$ is always positive. Therefore, the interval $(2;\ 4)$ also doesn't satisfy the inequality. Finally, crossing point 4, the expression changes its sign to '+', and therefore the interval $(4; 1)$ satisfies the inequality. So, the answer to the inequality is:

$$x \in (-\infty; -1] \cup 2 \cup [4; +\infty)$$

3. Solving rational inequalities:

Example 3. Solve the inequality $\frac{(x+1)(x-2)^2}{(x-4)^3} \geq 0$

Solution. Note that the factors in this inequality are exactly the same as in the previous example, so the solution will be the same as solving $(x + 1)(x - 2)^2(x - 4)^3 \geq 0$ with the small (but important) exception: the denominator cannot be equal to 0, and therefore, x cannot be equal to 4 — notice the round instead of square bracket in the answer! The x = 4 is a vertical asymptote

$$x \in (-\infty; -1] \cup 2 \cup (4;\ +\infty)$$

The x = 4 is a vertical asymptote; the function could go very very close to that value but can never have a value at 4.

4. Inequalities with absolute values (preview)

When you have an inequality with absolute value, you will have to consider two cases: when the expression under absolute value is positive and when the expression under the absolute value is negative – so use the definition of the absolute value:

$$|x| = \begin{cases} -x, & x < 0 \\ x, & x \geq 0 \end{cases}$$

Example 4. Solve inequality $|x - 4| < 7$.
Solution: consider the two cases; the one when the expression in the $|\ x - 4\ |$ is positive, $x - 4 \geq 0$ and the one when that expression is negative, $x - 4 < 0$.

Case 1. Set $x - 4 \geq 0$ means the inequality $|x - 4| < 7$ can be rewritten without the absolute value as $+(x - 4) < 7$. Solving this inequality gives us $x < 11$. But remember, x must be also greater than or equal to 4 (we started with this assumption) ! So, combining both inequalities together, we get $4 < x < 11$, or $x \in [4; 11)$.

$$\begin{cases} x - 4 \geq 0 \\ +(x - 4) < 7 \end{cases}$$

$$\begin{cases} x \geq 4 \\ x < 11 \end{cases}$$

Case 2. Set $\boldsymbol{x - 4 < 0}$. Then, the inequality can be rewritten without the absolute value sign as $\boldsymbol{-(x - 4) < 7}$. Solving this inequality gives us $x > -3$. But remember, x must also be less than 4! So, combining both things together, we get $-3 < x < 4$.

$$\begin{cases} (x - 4) < 0 \\ -(x - 4) < 7 \end{cases}$$

$$\begin{cases} x < 4 \\ x > -3 \end{cases}$$

Combining Cases 1 and 2 together, we get the final solution to the inequality: $-3 < x < 11$ or $x \in (-3; 11)$.

Homework problems

Instructions: Please always write solutions on a ***separate sheet of paper***. Solutions should include explanations. I want to see more than just an answer: I also want to see how you arrived at this answer, and some justification why this is indeed the answer. So **please include sufficient explanations**, which should be clearly written so that I can read them and follow your arguments.

1. Solve the following equations
   a. $|x-3|=5$
   b. $|2x-1| \ = \ 7$
   c. $|x^2-5|=4$

2. Solve the following equations. (Hint: you need to have common denominator first, then consider only the numerator and set it to zero. Remember, the denominator cannot be zero. Alternatively, cross multiply the terms to "remove" the denominator.)
   a. $\frac{(x+1)}{(x-1)} \ = \ 3$
   b. $\frac{(x^2-9)}{(x-1)} = (x+3)$
   c. $x \ - \ \frac{3}{x} \ = \ \frac{5}{x} \ - \ x$

3. Solve the following inequalities using **the snake** method. Show solutions on the real line and write the answers in the interval notation. (If you prefer, use the interval method instead)
   a. $(x-1)(x+2)>0$
   b. $(x+3)(x-2)^2 \leq 0$
   c. $x(x-1)(x+2) \geq 0$
   d. $x^2(x+1)^5(x+2)^3>0$

4. Solve the following inequalities, show solutions on the real line, and write the answers in the interval notation. Hint: convert this division into a multiplication problem.
   a. $\frac{x-2}{(x+3)} \leq 3$
   b. $(*) \ \frac{x(x-1)^2}{(x+1)^2} \geq \ 0$

5. Read the **4. Inequalities with absolute values (preview)** on page 2
   a. Solve again $|x-4|<7$
   b. $|x-2|>3$
   c. $|x-1|>x+3$