MATH 7
HANDOUT 14: VIETA FORMULAS AND QUADRATIC INEQUALITIES

A polynomial has the general form $p(x) = a_n x^n + a_{n-1}x^{n-1} + ... + a_2x^2 + a_1x + a_0$, where $a_n, a_{n-1}, ..., a_2, a_1, a_0$ are called coefficients. Examples of polynomials are: $x^5 + 2x^3 + x + 5$ or $2x^3 + 5x^2 + 3x + 1$. Notice that a quadratic is a polynomial where $n = 2$: $p(x) = a_2x^2 + a_1x + a_0$

Vieta formulas

If a polynomial $p(x)$ has a root r (i.e., if $p(r) = 0$), then $p(x)$ is divisible by $(x - r)$, i.e. $p(x) = (x - r)q(x)$ for some polynomial $q(x)$. In particular, if x_1, x_2 are roots of quadratic polynomial $ax^2 + bx + c$, then $ax^2 + bx + c = a(x - x_1)(x - x_2)$.

If we carry out the multiplication, we get $ax^2 + bx + c = a(x - x_1)(x - x_2) = a(x^2 - xx_2 - xx_1 + x_1x_2) = a(x^2 - x(x_1 + x_2) + x_1x_2)$ and

$$x_1 + x_2 = \frac{-b}{a}$$
$$x_1x_2 = \frac{c}{a}$$

In particular, if $a = 1$, then

$$x_1 + x_2 = -b$$
$$x_1x_2 = c$$

(Vieta formulas).

Solving polynomial inequalities

We discussed the general rule for solving polynomial inequalities:

- Find the roots and factor your polynomial, writing it in the form $p(x) = a(x - x_1)(x - x_2)$ (for polynomial of degree more than 2, you would have more factors).
- Roots $x_1, x_2, \dots$ divide the real line into intervals; define the sign of each factor and the product on each of the intervals.

Homework

1. Can you guess an analog of Vieta formulas for equation of degree 3: if x_1, x_2, x_3 are roots of an equation $x^3 + bx^2 + cx + d$, then what is the relation between b, c, d and x_1, x_2, x_3?

2. Let x_1, x_2 be roots of equation $x^2 + 5x - 7 = 0$. Find
(a) $x_1^2 + x_2^2$ (b) $(x_1 - x_2)^2$ (c) $\frac{1}{x_1} + \frac{1}{x_2}$ (d) $x_1^3 + x_2^3$
(hint for part (d): compute first $(x_1 + x_2)(x_1^2 + x_2^2)$)

***3.** Prove the statement we used in class: if a polynomial $p(x)$ has root r (i.e., if $p(r) = 0$), then $p(x)$ is divisible by $(x - r)$, i.e. $p(x) = (x - r)q(x)$ for some polynomial $q(x)$.

4. Solve the equation $x^4 - 3x^2 + 2 = 0$.

5. Solve the following equations and inequalities:

(a) $x^2 - 5x + 6 > 0$ (b) $x^2 < 1 + x$ (c) $\dfrac{x+1}{x-2} > 0$

(d) $x(x - 5)(x + 7) < 0$ (e) $\sqrt{2x + 1} = x$ (f) $\dfrac{2x+1}{x-3} > 1$

6. (a) Show that for any $a, b \geq 0$, one has $\frac{a+b}{2} \geq \sqrt{ab}$. (The left hand side is usually called the arithmetic mean of a, b; the right hand side is called the geometric mean of a, b.)
(b) Prove that for any $a > 0$, we have $a + \frac{1}{a} \geq 2$, with equality only when $a = 1$.

***7.** Solve equation $x^4 - 5x^3 + 6x^2 - 5x + 1 = 0$. [Hint: divide by x^2 and try to rewrite as an equation in $y = x + (1/x)$. The same trick works for any symmetric equation, in which coefficient of x^4 is the same as the constant term, and coefficient of x^3 is the same as coef. of x.]