MATH 5: HANDOUT 18
CHOOSINGS AND PERMUTATIONS.

CHOOSING WITH REPETITIONS, REVIEW

Here are basic combinatorics laws in one place for your convenience:

- Multiplication rule: if there are k ways to choose the first item, and n ways to choose the second, then there are $k \times n$ ways to choose the pair
- If we need to choose k items, each of which can be selected from a list of n, and **order matters, repetitions are allowed**, then there are n^k ways to do this.

- If we need to choose k items, each of which can be selected from a list of n, and **order matters, repetitions are not allowed**, there are $n(n-1)\dots(n-k+1)$ ways of doing it (the product has k factors). This number is usually denoted

$$ {}_kP_n = n(n-1)\dots(n-k+1) = \frac{n!}{(n-k)!}. $$

  Typical example: there are ${}_{10}P_{25}$ ways to seat 10 students in a room with 25 chairs.
- There are $k! = 1 \times 2 \times \cdots \times k$ ways to order k items.
  Typical example: there are $52!$ ways to shuffle a card deck.
- If we need to choose k items, each of which can be selected from a list of n, and **order does not matters, repetitions are not allowed**, then there are

$$ {}_kC_n = \frac{{}_kP_n}{k!} = \frac{n!}{k!(n-k)!} $$

  ways to do this.
  Typical examples: there are ${}_6C_{52}$ ways to choose six cards out of a deck of 52; if we toss a coin 10 times, there are ${}_4C_10$ combinations in which we have 4 heads and 6 tails.

Almost all combinatorial problems can be reduced to one of these.

IN CLASS PROBLEMS

1. Conor has 12 favorite books.

(a) He wants to put 6 of them in his backpack, and leave 6 at home. How many ways are there for him to do this?

Solutions: ${}_6C_{12} = 924$

(b) Now he wants to put some of the books in his backpack and leave the rest at home. How many ways are there to do this? ("Some" means anywhere from 0 to 12)

Solutions: He can put k on in bag and leave $12-k$ at home. Then the number of ways is

$$ \sum_{k=0}^{12} \binom{12}{k} = 2^{12} $$

We can think of this as the total number of $0/1$ states of the 12 books, 0 means stay at home 1 means go. This is 2^{12}.

2. Cristina also has 12 books.

(a) She wants to put them on two bookshelves she has, 6 books on one shelf and 6 on the other. How many ways are there for her to do this?

Solutions: ${}_6C_{12} \times 6!^2$

(b) She now wants to put 6 books in her backpack, and put the remaining books on the first bookshelf. How many ways are there for her to do this? (Note: for books on the shelf, the order matters; for books in the back pack, it does not.)

Solutions: ${}_6C_{12} \times 6!$

3. How many ways are there to put 10 books on two bookshelves? (The order on each shelf matters!)

Solutions: We can put k on one shelf, and $10 - k$ on the other. Then the number of ways is

$$\sum_{k=0}^{10} \binom{10}{k} k!(10-k)! = 10 \times 10!$$

4. How many "words" (or, rather combinations of letters — we do not care if they are meaningless) one can form by permuting letters in the word "problem"? In the word "paper"? In the word "letter"?

Solutions:

- For "problem", since all letters are distinct, we could have $7!$ total words.
- For "paper", there is one repeated letter. There are ${}_2C_5$ ways to place those two letters. Given their placement, there is $3!$ ways to arrange the other letters. In total ${}_2C_5 \times 3! = 60$ words. Another way to arrive at the answer is $5!/2! = 5 * 4 * 3 = 60$
- For "letter", there are 2 pairs of repeated letters. Thus, there are ${}_4C_6 = 15$ ways to position those, and ${}_2C_4 = 6$ arrangements in each such configuration. Thus in total there are ${}_4C_6 \times {}_2C_4 \times 2 = 180$. Another way to arrive at the answer is $6!/2!2! = 6 * 5 * 4 * 3/2 = 60$. In general, if you have n objects with r_1 of one kind, r_2 of another, $\ldots$, r_k of a kth kind, they can be arranged in

$$\frac{n!}{r_1! r_2! \ldots r_k!} \quad \text{ways.}$$

5. You toss a coin 16 times. What is the probability that you have no heads? Exactly one heads? That exactly half are heads?

Solutions: No heads is $\frac{1}{2^{16}}$, exactly one head is $\frac{1}{2^{16}} \times {}_1C_{16} = \frac{1}{2^{16}} \times 2^4 = \frac{1}{2^{12}}$. Exactly half are heads $\frac{1}{2^{16}} \times {}_8C_{16} = \frac{12870}{2^{16}} \approx 0.2$

6. You roll a die 10 times. What is the probability that all 10 will be sixes? That there will be no sixes? That exactly one will be a six? Exactly two?

Solutions: All 6s is $\frac{1}{6^{10}} \approx 1.6 \times 10^{-8}$. No 6s is $(\frac{5}{6})^{10} \approx .16$. Exactly 1 is ${}_1C_6 \times \frac{1}{6}(\frac{5}{6})^9 = (\frac{5}{6})^9 \approx 0.19$. Exactly 2 is ${}_2C_6 \times (\frac{1}{6})^2(\frac{5}{6})^8 = 15 \times \frac{1}{6^2}(\frac{5}{6})^8 \approx 0.096$