Classwork 21.

Irrational numbers

Rational number is a number which can be represented as a ratio of two integers:

$$a = \frac{p}{q}; \quad p \in Z, and\ q \in N, \quad (Z = \{\pm \cdots, \pm 1, 0\}, N = \{1, 2, \dots\})$$

Rational numbers can be represented as infinite periodical decimals (in the case of denominators containing only powers of 2 and 5 the periodical bloc of such decimal is 0).
Numbers, which can't be express as a ratio (fraction) $\frac{p}{q}$ for any integers p and q are irrational numbers. Their decimal expansion is not finite, and not periodical.
Examples:
0.01001000100001000001…
0.1234567891011121314151617181920 21…

What side the square with the area of a m^2 does have? To solve this problem, we have to find the number, which gives us a as its square. In other words, we have to solve the equation

$$x^2 = a$$

This equation can be solved (has a real number solution) only if a is nonnegative (($a \geq 0$) number. It can be seen very easily;

If $x = 0, \quad x \cdot x = x^2 = a = 0,$

If $x > 0, \quad x \cdot x = x^2 = a > 0,$

If $x < 0, \quad x \cdot x = x^2 = a > 0,$

We can see that the square of any real number is a nonnegative number, or there is no such real number that has negative square.

Square root of a (real nonnegative) number a is a number, square of which is equal to a.

There are only 2 square roots from any positive number, they are equal by absolute value, but have opposite signs. The square root from 0 is 0, there is no any real square root from negative real number.

Examples:

1. Find square roots of 16: 4 and (-4), $4^2 = (-4)^2 = 16$

2. Numbers $\frac{1}{7}$ and $\left(-\frac{1}{7}\right)$ are square roots of $\frac{1}{49}$, because $\frac{1}{7} \cdot \frac{1}{7} = \left(-\frac{1}{7}\right) \cdot \left(-\frac{1}{7}\right) = \frac{1}{49}$
3. Numbers $\frac{5}{3}$ and $\left(-\frac{5}{3}\right)$ are square roots of $\frac{25}{9}$, because $\left(\frac{5}{3}\right)^2 = \frac{5}{3} \cdot \frac{5}{3} = \left(-\frac{5}{3}\right)^2 = \left(-\frac{5}{3}\right) \cdot \left(-\frac{5}{3}\right) = \frac{25}{9}$

Arithmetic (principal) square root of a (real nonnegative) number a is a nonnegative number, square of which is equal to a.

There is a special sign for the arithmetic square root of a number a: $\sqrt{a}$.
Examples;

1. $\sqrt{25} = 5$, it means that arithmetic square root of 25 is 5, as a nonnegative number, square of which is 25. Square roots of 25 are 5 and (-5), or $\pm\sqrt{25} = \pm 5$
2. Square roots of 121 are 11 and (-11), or $\pm\sqrt{121} = \pm 11$
3. Square roots of 2 are $\pm\sqrt{2}$.
4. A few more:

$\sqrt{0} = 0$; $\sqrt{1} = 1$; $\sqrt{4} = 2$; $\sqrt{9} = 3$; $\sqrt{16} = 4$;

$\sqrt{25} = 5$; $\sqrt{\frac{1}{64}} = \frac{1}{8}$; $\sqrt{\frac{36}{25}} = \frac{6}{5}$

Base on the definition of arithmetic square root we can right

$$\left(\sqrt{a}\right)^2 = a$$

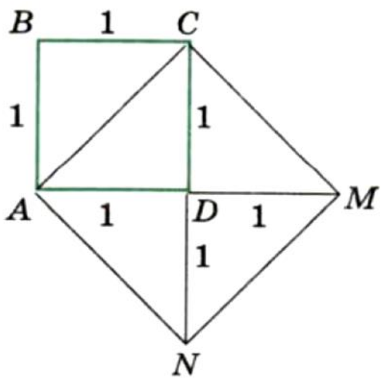
To keep our system of exponent properties consistent let's try to substitute $\sqrt{a} = a^k$. Therefore,

$$\left(\sqrt{a}\right)^2 = (a^k)^2 = a^1$$

But we know that

$$(a^k)^2 = a^{2k} = a^1 \quad \Rightarrow \quad 2k = 1, \; k = \frac{1}{2}$$

To solve equation $x^2 = 23$ we have to find two sq. root of 23. $x = \pm\sqrt{23}$. 23 is not a perfect square as 4, 9, 16, 25, 36 …



The length of the segment [AC] is $\sqrt{2}$ (from Pythagorean theorem). The area of the square ACMN is twice the area of the square ABCD. Let assume that the $\sqrt{2}$ is a rational number, so it can be represented as a ratio $\frac{p}{q}$, where $\frac{p}{q}$ is nonreducible fraction.

$$\left(\frac{p}{q}\right)^2 = 2 = \frac{p^2}{q^2}$$

Or $p^2 = 2q^2$, therefore p^2 is an even number, and p itself is an even number, and can be represented as $p = 2p_1$., consequently

$$p^2 = (2p_1)^2 = 4{p_1}^2 = 2q^2$$

$2{p_1}^2 = q^2 \Rightarrow q$ also is an even number and can be written as $q = 2q_1$.

$$\frac{p}{q} = \frac{2p_1}{2q_1}$$

therefore fraction $\frac{p}{q}$ can be reduced, which is contradict the assumption. We proved that the $\sqrt{2}$ isn't a rational number by contradiction.

$\sqrt{2}$ is an irrational number, therefore its decimal representation is an infinite nonperiodically decimal:

$$\sqrt{2} = a_0.a_1a_2 \dots$$

$$1 < 2 < 4; \quad \Rightarrow \quad \sqrt{1} < \sqrt{2} < \sqrt{4}; \qquad 1 < \sqrt{2} < 2; \qquad \Rightarrow \quad a_0 = 1$$

To find a_1 let's consider numbers $1.0,\ \ 1.1,\ \ 1.2,\ \ 1.3\ \ \dots$.

$$1.0^2 = 1; \quad 1.1^2 = 1.21; \quad 1.2^2 = 1.44$$

$$1.3^2 = 1.69; \quad 1.4^2 = 1.96; \quad 1.5^2 = 2.25$$

Therefore:

$$1.4^2 < 2 < 1.5^2, \quad 1.4 < \sqrt{2} < 1.5$$

$$\sqrt{2} = 1.4a_2a_3 \dots$$

To find the next digit,

$$1.40^2 = 1.96; \quad 1.41^2 = 1.9881; \quad 1.42^2 = 1.2.0164$$

$$\sqrt{2} = 1.41a_3 \dots; \quad \sqrt{2} = 1.4142135 \dots$$

Exercises:

1. Prove that the value of the following expressions is a rational number.

   a. $(\sqrt{2} - 1)(\sqrt{2} + 1)$

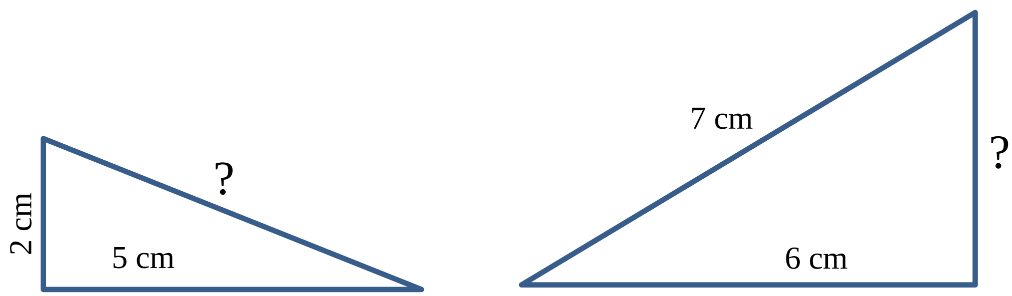
   b. $(\sqrt{5} - \sqrt{3})(\sqrt{5} + \sqrt{3})$

c. $\left(\sqrt{2}+1\right)^2+\left(\sqrt{2}-1\right)^2$
d. $\left(\sqrt{7}-1\right)^2+\left(\sqrt{7}+1\right)^2$
e. $\left(\sqrt{7}-2\right)^2+4\sqrt{7}$

2. Without using calculator compare:

$3\ \dots\ \sqrt{11}$	$11\ \dots\ \sqrt{110}$	$22\ \dots\ \sqrt{484}$
$5\ \dots\ \sqrt{20}$	$17\ \dots\ \sqrt{299}$	$35\ \dots\ \sqrt{1215}$

3. Find the missing length of the side of right triangles below:



4. Evaluate:

$a.\ 5\cdot\sqrt{4}\cdot 3;$ $\qquad b.\ 2\cdot\sqrt{9}+3\cdot\sqrt{16}$

$c.\ \sqrt{13-3\cdot 3};$ $\qquad d.\ \sqrt{7^2-26:2}$

$e.\ \frac{1}{2}\sqrt{5^2+22:2};$ $\qquad f.\ 3\sqrt{0.64}-5\cdot\sqrt{1.21}$