Math 5a. Classwork 8.

Rational numbers.

Positive rational number is a number which can be represented as a ratio of two natural numbers:

$$a = \frac{p}{q}; \qquad p, q \in N$$

As we know such number is also called a fraction, p in this fraction is a nominator and q is a denominator. Any natural number can be represented as a fraction with denominator 1:

$$b = \frac{b}{1}; \;\; b \in N$$

Basic property of fraction: nominator and denominator of the fraction can be multiplied by any non-zero number n, resulting the same fraction:

$$a = \frac{p}{q} = \frac{p \cdot n}{q \cdot n}$$

In the case that numbers p and q do not have common prime factors, the fraction $\frac{p}{q}$ is irreducible fraction. If $p < q$, the fraction is called “proper fraction”, if $p > q$, the fraction is called “improper fraction”.

If the denominator of fraction is a power of 10, this fraction can be represented as a finite decimal, for example,

$$\frac{37}{100} = \frac{37}{10^2} = 0.37, \qquad \frac{3}{10} = \frac{3}{10^1} = 0.3, \qquad \frac{12437}{1000} = \frac{12437}{10^3} = 12{,}437$$

$$10^n = (2 \cdot 5)^n = 2^n \cdot 5^n$$

$$\frac{2}{5} = \frac{2}{5^1} = \frac{2 \cdot 2^1}{5^1 \cdot 2^1} = \frac{4}{10} = 0.4$$

$$\begin{array}{r|l} & 0.875 \\ 8 & 7.000 \\ & -6.4 \\ & \;\;\;60 \\ & \;\;-56 \\ & \;\;\;\;\;40 \\ & \;\;\;-40 \\ & \;\;\;\;\;\;0 \end{array}$$

Therefore, any fraction, which denominator is represented by $2^n \cdot 5^m$ can be written in a form of finite decimal. This fact can be verified with the help of the long division, for example $\frac{7}{8}$ is a proper fraction, using the long division this fraction can be written as a decimal $\frac{7}{8} = 0.875$. Indeed,

$$\frac{7}{8} = \frac{7}{2 \cdot 2 \cdot 2} = \frac{7 \cdot 5 \cdot 5 \cdot 5}{2 \cdot 2 \cdot 2 \cdot 5 \cdot 5 \cdot 5} = \frac{7 \cdot 5^3}{2^3 5^3} = \frac{7 \cdot 125}{(2 \cdot 5)^3} = \frac{875}{10^3} = \frac{875}{1000} = 0.875$$

Also, any finite decimal can be represented as a fraction with denominator 10^n.

$$0.375 = \frac{375}{1000} = \frac{3}{8} = \frac{3}{2^3};$$

$$0.065 = \frac{65}{1000} = \frac{13 \cdot 5}{5^3 2^3} = \frac{13}{5^2 2^3};$$

$$6.72 = \frac{672}{100} = \frac{168}{25} = \frac{168}{5^2};$$

$$0.034 = \frac{34}{1000} = \frac{17 \cdot 2}{5^3 2^3} = \frac{17}{5^3 2^2};$$

$$\begin{array}{r} 0.71428571\ldots \\ 7\overline{)5.000} \\ -0\;0 \\ \hline 5\;0 \\ -4\;9 \\ \hline 10 \\ -\;7 \\ \hline 30 \\ -28 \\ \hline 20 \\ -14 \\ \hline 60 \\ -56 \\ \hline 40 \\ -35 \\ \hline 50 \\ -49 \\ \hline 10 \\ \ldots\ldots \end{array}$$

In other words, if the finite decimal is represented as an irreducible fraction, the denominator of this fraction will not have other factors besides 5^m and 2^n. Converse statement is also true: if the irreducible fraction has denominator which only contains 5^m and 2^n than the fraction can be written as a finite decimal. (Irreducible fraction can be represented as a finite decimal if and only if it has denominator containing only 5^m and 2^n as factors.)

If the denominator of the irreducible fraction has a factor different from 2 and 5, the fraction cannot be represented as a finite decimal. If we try to use the long division process, we will get an infinite periodic decimal.

At each step during this division, we will have a remainder. At some point during the process, we will see the remainder which occurred before. Process will start to repeat itself. On the example on the left, $\frac{5}{7}$, after 7, 1, 4, 2, 8, 5, remainder 7 appeared again, the fraction $\frac{5}{7}$ can be represented only as an infinite periodic decimal and should be written as $\frac{5}{7} = 0.\overline{714285}$. (Sometimes you can find the periodic infinite decimal written as $0.\overline{714285} = 0.(714285)$).

How we can represent the periodic decimal as a fraction?

Let's take a look on a few examples: $0.\overline{8}$, $2.35\overline{7}$, $0.\overline{0108}$.

$0.\overline{8}$. $x = 0.\overline{8}$ $10x = 8.\overline{8}$ $10x - x = 8.\overline{8} - 0.\overline{8} = 8$ $9x = 8$ $x = \frac{8}{9}$	$2.35\overline{7}$ $x = 2.35\overline{7}$ $100x = 235.\overline{7}$ $1000x = 2357.\overline{7}$ $1000x - 100x = 2357.\overline{7} - 235.\overline{7} = 2122$ $x = \frac{2122}{900} = \frac{1061}{450}$	$0.\overline{0108}$ $x = 0.\overline{0108}$ $10000x = 108.\overline{0108}$ $10000x - x = 108$ $x = \frac{108}{9999} = \frac{12}{1111}$

Now consider $2.4\overline{0}$ and $2.3\overline{9}$

$x = 2.4\overline{0}$
$10x = 24.\overline{0}$
$100x = 240.\overline{0}$

$100x - 10x = 240 - 24$
$x = \frac{240 - 24}{90} = \frac{216}{90} = 2.4$

$x = 2.3\overline{9}$
$10x = 23.\overline{9}$
$100x = 239.\overline{9}$

$100x - 10x = 239 - 23$
$x = \frac{239 - 23}{90} = \frac{216}{90} = 2.4$

Algebraic expression.

Expressions where variables, and/or numbers are added, subtracted, multiplied, and divided.

For example:

$$2a; \quad 3b + 2; \quad 3c^2 - 4xy^2$$

We can do a lot with algebraic expressions, even so we don't know exact values of variables. First, we always can combine like terms:

$$2x + 2y - 5 + 2x + 5y + 6 = 2x + 2x + 5y + 2y + 6 - 5 = 4x + 7y + 1$$

We can multiply an algebraic expression by a number or a variable:

$$3 \cdot (1 + 3y) = 3 \cdot 1 + 3 \cdot 3y = 3 + 9y$$

In this example the distributive property was used. Using the definition of multiplication we can write:

$$3 \cdot (1 + 3y) = (1 + 3y) + (1 + 3y) + (1 + 3y) = 3 + 3 \cdot y = 3 + 9y$$

Another example:

$$5a(5 - 5x) = \underbrace{(5 - 5x) + (5 - 5x) + \cdots + (5 - 5x)}_{5a\ times} = \underbrace{5 + 5 + \cdots + 5}_{5a\ times} - \underbrace{5x - 5x - \cdots - 5x}_{5a\ times}$$

$$= \underbrace{5 + 5 + \cdots + 5}_{5a\ times} - \underbrace{5x - 5x - \cdots - 5x}_{5a\ times} = 5a \cdot 5 - 5a \cdot 5x = 25a - 25ax$$

If we need to multiply two expressions

$$(a + 2) \cdot (a + 3)$$

We can use a substitution technic, we will substitute one of the expressions with a variable, for example, instead of $(a + 2)$ we can use u.

$$(a + 2) = u$$

And then we will multiply

$$u \cdot (a + 3) = u \cdot a + 3u$$

We know, that actually u should not be there, $(a+2)$ should.

$$u \cdot (a+3) = (a+2)\ \cdot a + 3(a+2)$$

We know how to multiply an expression by a variable (or number):

$$(a+2)\cdot a + 3(a+2) = a\cdot a + 2a + 3a + 3\cdot 2 = a^2 + 5a + 6$$

$$(a+2)\cdot(a+3) = a\cdot a + 3a + 2a + 3\cdot 2 = a^2 + 5a + 6$$

$$(a+2)\cdot(a+3) = a^2 + 5a + 6$$

There are a few very useful products:

$$(a+b)^2 = (a+b)\cdot(a+b) = a\cdot a + a\cdot b + b\cdot a + b\cdot b = a^2 + 2ab + b^2$$

Let's do a few examples:

$$(2+x)^2 = (2+x)(2+x) = 2\cdot 2 + 2\cdot x + x\cdot 2 + x\cdot x = 2^2 + 2x + 2x + x^2 = x^2 + 2\cdot 2x + 4$$
$$= x^2 + 4x + 4$$

$$(ab+2y)^2 = (ab+2y)(ab+2y) = ab\cdot ab + ab\cdot 2y + 2y\cdot ab + 2y\cdot 2y$$
$$= a^2b^2 + 2yab + 2yab + 4y^2 = a^2b^2 + 4yab + 4y^2$$

$$(a-b)(a+b) = a\cdot a + a\cdot b - a\cdot b + b\cdot b = a^2 - b^2$$

Exercises.

1. Evaluate the following using decimals:

$a.\ 0.36 + \frac{1}{2};$ $\quad b.\ 5.8 - \frac{3}{4};$ $\quad c.\ \frac{2}{5} : 0.001;$ $\quad d.\ 7.2 \cdot \frac{1}{1000}$

2. Evaluate the following using fractions:

$a.\ \frac{2}{3} + 0.6;$ $\quad b.\ 1\frac{1}{6} - 0.5;$ $\quad c.\ 0.3 \cdot \frac{5}{9};$ $\quad d.\ \frac{8}{11} : 0.4;$

3. Write as a fraction

$a.\ 0.\overline{5},$ $\quad b.\ 0.5,$ $\quad c.\ 0.\overline{7},$ $\quad d.\ 0.7,$ $\quad e.\ 0.1\overline{2},$ $\quad f.\ 0.\overline{12},$ $\quad g.\ 0.12$

4. Multiply.

$a.\ (a+2)(a+2);$ $\quad f.\ (a+1)(a+3);$

$b.\ (3+y)(y+4);$ $\quad g.\ (c+d)(c-2d);$

$c.\ (3+x)(3-x);$ $\quad h.\ (y-2)(3-y);$

$d.\ (x-y)(x+y);$ $\quad i.\ (x-m)(x-m);$

$e.\ (2a+c)(a+ac);$ $\quad j.\ (2d+3l)(2d+3l)$