MATH 10
ASSIGNMENT 23: SIGN OF A PERMUTATION
APRIL 7, 2024

Definition. Let f be a permutation of $\{1, \dots, n\}$. An **disorder for** f is a pair i, j such that $i < j$ but $f(i) > f(j)$. For example, for permutation

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 4 & 3 & 1 \end{pmatrix}$$

there are 4 disorders: $(1, 2)$, $(1, 3)$, $(1, 4)$, $(3, 4)$.

A *sign* of a permutation is defined by

$$\operatorname{sgn}(f) = (-1)^{\#\text{ of disorders}}$$

thus, $\operatorname{sgn}(f) = +1$ if the number of disorders is even (such permutations are called *even*), and $\operatorname{sgn}(f) = -1$ if the number of disorders is odd (such permutations are called *odd*).

1. Is the cycle of length n even or odd?

2. For any permutation $s \in S_n$ and a polynomial p in variables $x_1, \dots, x_n$, we can define new polynomial $s(p)$ by permuting $x_1, \dots, x_n$ using s. For example, if $p = x_1^2 + 2x_2 + x_1x_3$, and $s = (1\,2)$, then $s(p) = x_2^2 + 2x_1 + x_2x_3$.

(a) Show that for the polynomial in 3 variables $p = (x_1 - x_2)(x_1 - x_3)(x_2 - x_3)$, and any permutation s, we have $s(p) = \operatorname{sgn}(s) \cdot p$.

(b) Can you construct a polynomial p in n variables such that $s(p) = \operatorname{sgn}(s) \cdot p$ for any permutation $s \in S_n$?

3. (a) Show that if $s \in S_n$ is even (respectively odd) then $(i\,i+1) \circ s$ is odd (respectively, even). Here $(i\,i+1)$ is a transposition which exchanges numbers i and $i+1$. [Hint: this transposition changes the order of exactly one pair.]

(b) Show that if s is even (respectively odd) and τ is any transposition, then $\tau \circ s$ is odd (respectively, even).

(c) Show that s is even if and only if it can be written as a product of even number of transpositions.

4. Show that for any permutations $s, t \in S_n$, we have $\operatorname{sgn}(st) = \operatorname{sgn}(s)\operatorname{sgn}(t)$.