MATH 10
ASSIGNMENT 22: PERMUTATIONS
MARCH 24, 2024

A **permutation** of some set S is a function $f\colon S \to S$ which is a bijection (one-to-one and onto, or invertible). We will only be discussing permutations of finite sets, usually the set $S = \{1, \dots, n\}$. In this case one can also think of a permutation as a way of permuting n items placed in boxes labeled $1, \dots, n$: namely, move item from box 1 to box $f(1)$, item from box 2 to $f(2)$, etc. The set of all permutations of $\{1, \dots, n\}$ is denoted by S_n.

Notation: the permutation f which sends 1 to a_1, 2 to a_2, etc, is usually written as

$$\begin{pmatrix} 1 & 2 & \dots & n \\ a_1 & a_2 & \dots & a_n \end{pmatrix}$$

An alternative way of describing a permutation is by using the notion of cycles. A **cycle** $(a_1 a_2 \dots a_k)$ is a permutation which sends a_1 to a_2, a_2 to a_3, …, a_n to a_1 (and leaves all other elements unchanged). For example, (123) is the permutation such that $f(1) = 2$, $f(2) = 3$, $f(3) = 1$ and $f(a) = a$ for all other a.

Permutations can be composed in the usual way: $f \circ g(x) = f(g(x))$. This operation is associative but in general not commutative: $f \circ g \neq g \circ f$.

1. How many permutations of the set $\{1, \dots, n\}$ are there?

2. Compute the following compositions (a) $(12) \circ (13)$ (b) $(12) \circ (23)$ (c) $(23) \circ (12)$ (d) $(12) \circ (13) \circ (12)$ (e) $(123) \circ (132)$ (f) $(38) \circ (123456) \circ (38)$

3. Find the inverse of permutation

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 3 & 1 & 6 & 4 & 2 & 5 \end{pmatrix}$$

4. Show that for permutations s_1, s_2 we have $(s_1 s_2)^{-1} = s_2^{-1} s_1^{-1}$.

5. Fifteen students are meeting in a classroom which has 15 chairs numbered 1 through 15. The teacher requires that every minute they change seats following this rule:

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
3	5	10	8	11	14	15	6	13	1	4	9	7	2	12

(e.g., the student who was sitting in the chair number 1 would move to chair number 3).
In how many minutes will the students return to their original seats?

6. (a) Let a permutation f be a product of non-intersecting cycles of lengths $n_1, n_2, \dots, n_l$ (in this case, we will say that it has the **type** $\langle n_1, n_2, \dots n_l \rangle$). What is the order of f, i.e. the smallest d such that $f^d = id$, where id is the identity permutation: $id(a) = a$?
(b) Find permutations of the set $\{1, \dots, 9\}$ which have orders 7, 10, 12, 11.

7. (a) Write the permutations in problems 3, 4 as products of non-intersecting cycles.
(b) Show that any permutation can be written as a product of non-intersecting cycles.

8. A transposition is a permutation that exchanges two numbers, leaving all others in place — e.g. (12) or (57).
(a) Consider this sequence of numbers: $4, 2, 1, 3, 5, 3$ (obviously, it is obtained from sequence $1, 2, 3, 4, 5$ by a permutation s)
Show how one can put them in a correct order by applying a sequence of transpositions, each exchanging two adjacent numbers.
(b) Show how one can write the permutation s of the previous part as a product of transpositions of the form $(ii+1)$.
(c) Show that any permutation can be written as a product of transpositions of the form $(ii+1)$.

9. (a) Write the permutation $\begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix}$ as a product of transpositions of the form $(i\ i+1)$ in 2 different ways.
(b) Can you write it as a product of even number of transpositions?