MATH 10
ASSIGNMENT 13: OPEN AND CLOSED SETS
JAN 7, 2024

Definition 1. A metric space is a set X with a distance function: for any $x, y \in X$ we have a real number $d(x, y)$ such that

1. $d(x, y) = d(y, x)$
2. $d(x, y) \geq 0$ for any x, y
3. $d(x, y) = 0$ if and only if $x = y$
4. Triangle inequality: $d(x, y) + d(y, z) \geq d(x, z)$.

Usual examples are $\mathbb{R}$, $\mathbb{R}^2$, ..., but there are other examples as well.
Given a point $x \in X$ and a positive real number ε, we define ε-neighborhood of x by

$$B_\varepsilon(x) = \{y \in X \mid d(x, y) < \varepsilon\}.$$

If $S \subset X$, denote by S' the complement of S. Then, for any $x \in X$, we can have one of three possibilities:

1. There is a neighborhood $B_\varepsilon(x)$ which is completely inside S (in paritcular, this implies that $x \in S$). Such points are called *interior points* of S; set of interior points is denoted by $\mathrm{Int}(S)$.
2. There is a neighborhood $B_\varepsilon(x)$ which is completely inside S' (in particular, this implies that $x \in S'$). Thus, $x \in \mathrm{Int}(S')$.
3. Any neighborhood of x contains points from S and points from S' (in this case, we coudl have $x \in S$ or $x \in S'$). Set of such points is called the *boundary* of S and denoted ∂S.

Definition 2. A set S is called *open* if every point $x \in S$ is an interior point: $S = \mathrm{Int}(S)$.
A set S is called closed if $\partial S \subset S$.

Homework

1. Show that set $\mathbb{R}^2$ with distance defined by

$$d((x_1, y_1), (x_2, y_2)) = |x_1 - x_2| + |y_1 - y_2|$$

is a metric space. (This distance is sometimes called *Manhattan* or taxicab distance — can you guess why?)

2. For each of the following subsets of $\mathbb{R}$, find its interior and boundary and determine if it is open, closed, or neither.
(a) Set $\mathbb{N} = \{1, 2, 3, \dots\}$.
(b) Interval $[0, 1]]$
(c) Open interval $(0, 1)$
(d) Interval $[0, 1)$.
(e) Set of all rational numbers
(f) Set consisting of just two points $\{0, 1\}$
*(g) Set $x^3 + 2x + 1 > 0$
Are there any subsets of $\mathbb{R}$ which are both open and closed?

3. Let X be an arbitrary metric space, and let $p \in X$. For each of the following subsets of X, find its interior and boundary, and determine if it is open, closed, or neither.
(a) Open ball $B_r(p) = \{x \mid d(x, p) < r\}$.
(b) Closed ball $\overline{B}_r(p) = \{x \mid d(x, p) \leq r\}$.

4. Show that a set S is open if and only if its complement S' is closed.

5. Show that union and intersection of two open sets is open. Is it true if instead of two sets we consider any collection (possibly infinite) of open sets?
Same question about closed sets.

***6.** For a set S, let $\overline{S} = S \cup \partial S = \{x \mid \text{ In any neighborhood of } x\text{, there are elements of } S\}$. This set is called the closure of S; motivation for this name will become clear in a second.

(a) Show that S is closed if and only if $\overline{S} = S$.

(b) Consider the set $\overline{\overline{S}}$ (note two bars) – closure of closure of S. Show that for any point $p \in \overline{\overline{S}}$, there is a point from S within distance 1 from p.

(c) Show that $\overline{\overline{S}} = \overline{S}$; deduce from it that $\overline{S}$ is closed. (This explains the name closure)

***7.** Consider the set P of all polynomials in variable x with real coefficients. Define the **norm** of a polynomial by

$$\|p\| = 2^{-n} \qquad \text{if } p(x) \text{ has root of multiplicity } n \text{ at } x = 0$$

(i.e., if $p(x) = x^n(a_0 + a_1x + \dots), \quad a_0 \neq 0$).

For example, for any polynomial that has non-zero constant term, $\|p\| = 1$; for a polynomial which has zero as a root of multiplicity 1, $\|p\| = 2^{-1}$.

Define now distance between two polynomials by

$$f(f, g) = \|f - g\|.$$

For example, distance between $1+x+x^2+x^7$ and $1+x+x^3$ is 2^{-2}, since their difference is $x^2-x^3+x^7$.

Prove that this satisfies the properties of distance; moreover, in this case we have a stronger condition: given 3 polynomials f, g, h, we have

$$d(f, h) \leq \max(d(f, g), d(g, h))$$