MATH 10
ASSIGNMENT 10: VECTOR SPACES AND DIMENSION

DECEMBER 10, 2023

Review of last time

Recall that we can systematically represent a system of linear equations

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n &= b_2 \\ &\vdots \\ a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n &= b_m \end{aligned} \tag{1}$$

in terms of its augmented matrix

$$A|\mathbf{b} = \begin{bmatrix} a_{11} & a_{12} & \ldots & a_{1n} & | & b_1 \\ a_{21} & a_{22} & \ldots & a_{2n} & | & b_2 \\ \vdots & \vdots & \ddots & \vdots & | & \vdots \\ a_{m1} & a_{m2} & \ldots & a_{mn} & | & b_m \end{bmatrix}.$$

Using elementary row transformations, such an augmented matrix can be brought to the 'row echelon form", where each row begins with some number of zeroes, and each next row has more zeroes than the previous one:

$$\begin{bmatrix} X & * & * & * & * & * & * & | * \\ 0 & 0 & 0 & X & * & * & * & | * \\ 0 & 0 & 0 & 0 & X & * & * & | * \end{bmatrix}$$

(here X's stand for non-zero entries).

To solve such a system, we do the following:

- Variables corresponding to columns with X's in them are called pivot variables; the remaining ones are called free variables.
- Values for free variables can be chose arbitrarily. Values for pivot variables are then uniquely determined from the equations.

For example, in the system

$$\begin{aligned} x_1 + x_2 + x_3 &= 5 \\ x_2 + 3x_3 &= 6 \end{aligned} \tag{2}$$

varaibles x_1, x_2 are pivot, and variable x_3 is free, so we can solve it by letting $x_3 = t$, and then

$$\begin{aligned} x_2 &= 6 - 3x_3 = 6 - 3t \\ x_1 &= 5 - x_2 - x_3 = -1 + 2t \end{aligned}$$

Systems with no solutions

It could happen that a system of linear equations has no solutions. For example, if the augmented matrix is

$$\begin{bmatrix} 1 & 1 & | \, 2 \\ 0 & 0 & | \, 1 \end{bmatrix}$$

then the second equation reads $0 \cdot x_1 + 0 \cdot x_2 = 1$, which clearly has no solutions. It happens if in the row echelon form, there is an row which has all zero entries except the last one (corresponding to the right hand side of the equation), which is non-zero.

Lines and planes in $\mathbb{R}^3$

As we saw before, a single linear equation $ax_1+bx_2+cx_3=d$ describes a plane perpendicular to the vector (a,b,c) in $\mathbb{R}^3$ (we assume that at least one of a,b,c is non-zero). If we use our system of linear equation techniques to solve this equation, we obtain something of the form

$$\mathbf{x}=\mathbf{a}+t_1\mathbf{v}_1+t_2\mathbf{v}_2,$$

where $t_1,t_2\in\mathbb{R}$ are parameters which can take any real values and $\mathbf{a},\mathbf{v}_1,\mathbf{v}_2\in\mathbb{R}^3$ are some fixed vectors found by solving the equation. This is the *parametric* definition of a plane. Note that we need 2 parameters, which of course is related to the fact that a plane is 2-dimensional — more on this below.

Likewise, the parametric definition of a line in three-dimensional space is

$$\mathbf{x}=\mathbf{a}+t\mathbf{v},$$

for some vectors $\mathbf{a},\mathbf{v}\in\mathbb{R}^3$.

Matrices and vectors

We will write elements of $\mathbb{R}^n$ as columns:

$$\mathbf{x}=\begin{bmatrix}x_1\\x_2\\\vdots\\x_n\end{bmatrix}$$

Given such a vector $\mathbf{x}$ and a matrix A with m rows and n columns, we define the new vector $A\mathbf{x}\in\mathbb{R}^m$ by

$$A\mathbf{x}=\begin{bmatrix}a_{11}x_1+a_{12}x_2+\cdots+a_{1n}x_n\\a_{21}x_1+a_{22}x_2+\cdots+a_{2n}x_n\\\cdots\\a_{m1}x_1+a_{m2}x_2+\cdots+a_{mn}x_n\end{bmatrix}$$

Thus, we can write system (1) as $A\mathbf{x}=\mathbf{b}$.

Spaces of solutions

We say that a subset V of $\mathbb{R}^n$ is a *s*ubspace if for any $\mathbf{v},\mathbf{v}'\in V$ and a real number c, we have $c\mathbf{v}\in V$ and $\mathbf{v}+\mathbf{v}'\in V$ (in particula,r it means that $0\in V$. For example, subset of $\mathbb{R}^3$ given by equation $x_1+x_2=0$ is a subspace. More generally, for any equation of the form $A\mathbf{x}=0$, the space of solutions is a subspace of $\mathbb{R}^n$.

We say that subspace V has dimension d if one can find d vectors $\mathbf{v}_1,\dots\mathbf{v}_d$ in V such that

1. Every vector in V can be written in the form

$$\mathbf{x}=t_1\mathbf{v}_1+\cdots+t_d\mathbf{v}_d$$

for some $t_1,\dots,t_n\in\mathbb{R}$. (Expressions of this form are called *l*inear combinations of vectors $\mathbf{v}_1,\dots,\mathbf{v}_d$.)

2. Vectors $\mathbf{v}_1,\dots,\mathbf{v}_d$ are independent: no one of them can be written as a combination of others.

In this case, the collection $\mathbf{v}_1,\dots,\mathbf{v}_d$ is called a *b*asis of V.

For example, the space $\mathbb{R}^n$ itself has dimension n.

More generally, if V is the set of vectors $\mathbf{a}+t_1\mathbf{v}_1+\cdots+t_n\mathbf{v}_n$, and vectors $\mathbf{v}_1,\dots,\mathbf{v}_n$ are independent, we say that V has dimension d (in this siutation, V is called an affine subspace — which just means a usual subspace but translated to away from origin).

Theorem. *The dimension of the space of solutions of a system of linear equations $A\mathbf{x}=0$ is given by*

$$d=(\textit{number of variables})-(\textit{number of nonzero rows in row echelon form})$$

More generally, for system of equations $A\mathbf{x}=\mathbf{b}$, if it has any solutions at all, then the dimension of the seet of solutions is given by the same formula as above.

Indeed, the number of free variables is the number of all variables minus the number of pivot variables.

Thus, typically we expect that a system with n variables and k equations has $n-k$ dimensional space of solutions. This is not always true: it could happen that after bringing it to row echelon form, some rows become zero, or that the system has no solutions at all — but these situations are unusual (at least if $k\le n$).

Homework

1. Consider the system of equations $A\mathbf{x} = 0$, where $\mathbf{x} \in \mathbb{R}^4$ and the matrix A is given by

$$A = \begin{bmatrix} -3 & -4 & -12 & 1 \\ 4 & 4 & 12 & 0 \\ -11 & -12 & -35 & 1 \end{bmatrix}$$

Bring it to row echelon form and write the most general solution. What is the dimension of space of solutions?

2. Write the equation of a plane in $\mathbb{R}^3$ passing through the points $(1,0,0)$, $(0,1,0)$, $(0,0,1)$. [Hint: if the equation of the plane is $ax_1 + bx_2 + cx_3 = d$, then plugging in it each of the points gives a condition on a, b, c, d; this gives a system of linear equations.]

Is such a plane unique?

3. Let P be the plane in $\mathbb{R}^3$ described parametrically as the set of all points of the form

$$\mathbf{x} = t_1 \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} + t_2 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \qquad t_1, t_2 \in \mathbb{R}.$$

Write an equation of this plane. [Hint: similar to previous problem.]

4. Consider two planes $2x + 3y - z = 0$, $4x + z = 4$. Prove that their intersection is a line; write the line in the parametric form, by writing a generic point in the line as $\mathbf{x} = \mathbf{a} + t\mathbf{v}$ for some vector $\mathbf{v}$.

5. Find the intersection of 3 planes

$$\begin{aligned} x + 2y + 3z &= 3 \\ 3x + y + 2z &= 3 \\ 2x + 3y + z &= 3 \end{aligned}$$

6. (a) Prove that if we have a system of linear equations $A\mathbf{x} = 0$, where $\mathbf{x} \in \mathbb{R}^n$, and the number of equations m is less than n, then it must have at least one non-zero solution.

(b) Show that any collection of vectors $\mathbf{v}_1, \dots, \mathbf{v}_n$ in $\mathbb{R}^m$ is dependent: one can find numbers $t_1, \dots, t_n$, not all zero, such that $t_1\mathbf{v}_1 + \cdots + t_n\mathbf{v}_n = 0$. [Hint: this can be considered as a system of linear equations on t_i.]