MATH 10
ASSIGNMENT 5: DOT PRODUCTS
OCTOBER 22,2023

Dot product

By the Pythageorean theorem, for a vector $\mathbf{v} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$, its length is given by $\sqrt{x^2+y^2+z^2}$. It is common to denote the length of a vector $\mathbf{v}$ by $|\mathbf{v}|$:

$$|\mathbf{v}| = \sqrt{x^2+y^2+z^2}.$$

A convenient tool for computing lengths is the notion of the *dot product*. The dot product of two vectors is a number (not a vector!) defined by

$$\begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix} \bullet \begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix} = x_1x_2 + y_1y_2 + z_1z_2.$$

The dot product has the following properties:

1. It is symmetric: $\mathbf{v}\bullet\mathbf{w} = \mathbf{w}\bullet\mathbf{v}$

2. It is linear as function of $\mathbf{v}$, $\mathbf{w}$:

$$(\mathbf{v}_1+\mathbf{v}_2)\bullet\mathbf{w} = \mathbf{v}_1\bullet\mathbf{w} + \mathbf{v}_2\bullet\mathbf{w}$$
$$(c\mathbf{v})\bullet\mathbf{w} = c(\mathbf{v}\bullet\mathbf{w})$$

3. $\mathbf{v}\bullet\mathbf{v} = |\mathbf{v}|^2$, or, equivalently, $|\mathbf{v}| = \sqrt{\mathbf{v}\bullet\mathbf{v}}$

4. Vectors $\mathbf{v}$, $\mathbf{w}$ are perpendicular iff $\mathbf{v}\bullet\mathbf{w} = 0$.

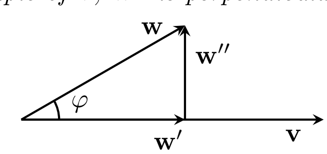
The first three properties are immediate from the definition. The last one follows from the Pythagorean theorem: if $\mathbf{v} \perp \mathbf{w}$, then by Pythagorean theorem, $|\mathbf{v}|^2 + |\mathbf{w}|^2 = |\mathbf{v}-\mathbf{w}|^2 = (\mathbf{v}-\mathbf{w})\bullet(\mathbf{v}-\mathbf{w}) = \mathbf{v}\bullet\mathbf{v} + \mathbf{w}\bullet\mathbf{w} - 2\mathbf{v}\bullet\mathbf{w}$.

From these properties one easily gets the following important result:

Theorem.

$$\mathbf{v}\bullet\mathbf{w} = \mathbf{v}\bullet\mathbf{w}' = |\mathbf{v}|\cdot|\mathbf{w}|\cos\varphi$$

where φ is the angle between vectors $\mathbf{v}$, $\mathbf{w}$, and $\mathbf{w}'$ is projection of $\mathbf{w}$ onto $\mathbf{v}$. It can be defined by saying that $\mathbf{w} = \mathbf{w}' + \mathbf{w}''$, and vector $\mathbf{w}'$ is a multiple of $\mathbf{v}$, $\mathbf{w}''$ is perpendicular to $\mathbf{v}$:



This theorem is commonly used to find the angle between two vectors:

$$\cos\varphi = \frac{\mathbf{v}\bullet\mathbf{w}}{|\mathbf{v}|\cdot|\mathbf{w}|}$$

Homework

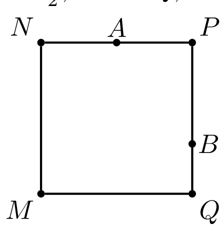
In all the questions where they ask you to find an angle, an answer like $\sin^{-1}(1/3)$ is perfectly acceptable — you do not have to compute approximate value. However, if there is a way to simplify the answer, please do it (e.g. simplifying $\sin^{-1}(1/2)$ to $\pi/6 = 30°$).

1. Prove that the triangle with vertices at $A(3,0)$, $B(1,5)$, and $C(2,1)$ is obtuse. Find the cosine of the obtuse angle.

2. Prove the law of cosines: in a triangle $\triangle ABC$, with sides $AB = c$, $AC = b$, $BC = a$, one has $c^2 = a^2 + b^2 - 2ab\cos\angle C$. [Hint: $c^2 = \overrightarrow{AB} \bullet \overrightarrow{AB}$, and $\overrightarrow{AB} = \overrightarrow{CB} - \overrightarrow{CA}$.]

3. (a) Let $\mathbf{v}$, $\mathbf{w}$ be two vectors in the plane. Show that if $|\mathbf{v}| = |\mathbf{w}|$, then $(\mathbf{v} - \mathbf{w}) \cdot (\mathbf{v} + \mathbf{w}) = 0$.
(b) Use the previous part to show that the two diagonals of a rhombus are perpendicular.

4. On the sides of a square $MNPQ$, with side 1, the points A and B are taken so that $A \in NP$, $NA = \frac{1}{2}$, $B \in PQ$, and $QB = \frac{1}{3}$. Prove that $\angle AMB = 45°$.



5. Use dot product to find the angle between two diagonals of a unit cube. (You will need to first write each diagonal in coordinates, as a vector.)

6. A billiard ball traveling with velocity $\vec{v}$ hits another ball which was at rest. After the collision, balls move with velocities $\vec{v}_1$, $\vec{v}_2$. Prove that $\vec{v}_1 \perp \vec{v}_2$, using the following conservation laws (m is the mass of each ball which is supposed to be the same)

Momentum conservation: $m\vec{v} = m\vec{v}_1 + m\vec{v}_2$

Energy conservation: $\frac{m|\vec{v}|^2}{2} = \frac{m|\vec{v}_1|^2}{2} + \frac{m|\vec{v}_2|^2}{2}$

7. Consider the plane given by equation $ax + by + cz = d$.
(a) Let $P_1 = (x_1, y_1, z_1)$, $P_2 = (x_2, y_2, z_2)$ be two points on this plane. Prove that then

$$a(x_1 - x_2) + b(y_1 - y_2) + c(z_1 - z_2) = 0.$$

(b) Prove that $\overrightarrow{P_1P_2}$ is perpendicular to vector $\mathbf{v} = (a, b, c)$.
(In such a situation — when any vector contained in the plane is perpendicular to $\mathbf{v}$ — we say the plane is perpendicular to $\mathbf{v}$.)