MATH GAMES

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Game Theory

In all of the games we will be dealing with there are 2 players. Because these are mathematical games, one of the players always has a winning strategy. Your goal is to figure out who has the winning strategy and what is.

A winning strategy is an algorithm which guarantees a win for the corresponding player REGARDLESS of his opponent's moves. It must be accompanied by a proof of why this algorithm is indeed a winning one. Argument "I tried it several times and it seems to work" is not enough.

Here are some ideas you can use for constructing such a strategy:

1. **Winning/Losing Position:** A winning position is a position such that if a player is in this position (before his move), he is guaranteed to win the game. The most important idea here is that if a player is in a certain position and regardless of which move he makes, his opponent will be in a winning position, then that certain position is a losing position. On the other hand, if a player is in a certain position, and can always make a move so that his opponent will be in a losing position, then that certain position is a winning position.
2. **Work Backwards:** Think about when the game ends. What must be the move(s) that happened right before the game ends? You can figure out some winning and losing positions, and from there work your way backwards to figure out if the starting position is winning or losing.
3. **Symmetry** A very simple strategy is to copy or almost copy your opponent's moves. Here is an example. Two players take turns placing identical coins on a round table. The coins cannot overlap. A player loses if they can't make a move. Who can guarantee to win?

   Solution: The first player should place a coin in the exact center of the table. Then she places every next coin in the position that is symmetric with respect to the center to the position of the last coin placed by her opponent.

Your task is analyzing each of the games below to find a winning strategy for one of the players. Each is played by two players.

Some simple games

1. Two players are moving the hour hand of the clock. Each player can move it forward (clockwise) either 2 or 3 hours. Initially the hand points to 1; the player who moves it to 12 wins.

2. Two brothers collected candy on Halloween and split it into two piles, one with 18 pieces, the other, with 23. They decided to play the following game: at each turn, you can eat all candy from one of the piles and divide the other into two new piles. If you can't do it, you lose.

   Which of the brothers will win this game?

3. On one square of an 8 by 8 chessboard there is a "lame tower" that can move either to the right or up by any number of squares.Two players take turns moving the tower. The player unable to move the tower loses. (Consider various initial positions of the tower.)

4. Two players are placing X and O in the squares of $N \times 1$ strip of paper. First player uses X, the second O. The only rule is that you are not allowed to have two X next to each other and not allowed to have two O next to each other. If you can't make a move, you lose.

   Which player has a winning strategy?

5. There are 5 buckets arranged in a row. Every bucket contains a marble. On every move a player can select one of the buckets (except the rightmost bucket) and move all the marbles in the bucket to the neighboring bucket to the right. The player who cannot make a move loses.

6. A 1001 step staircase is leading to the summit of a mountain. On some of the steps, there are stones. Sisyphus and Ares take turns moving these stones.

   Sisyphus can take any stone and take it up to the nearest empty step.

Ares can take any stone such that the step immediately below is empty, and roll that stone down to that step.

Sisyphus' goal is to place a stone on the top step; Ares' goal is to not allow Sisyphus to do that.

If there are 500 stones which initially were placed on steps 1–500, and Sisyphus makes the first move, can he win?

More advanced games

8. Two boys took a dog for walk in the park. The park has the shape of a square, with 4 alleys along the sides and 2 more alleys connecting middles of opposite sides of the park.

The dog ran away from the boys. Can they catch him if the dog runs 3 times as fast as either of the boys?

They are only allowed to run along the alleys, and the boys can hear dog's barking all the time so they know where it is.

9. Same question as in the previous problem, but now the park has the shape of an equilateral triangle, with the alleys along the sides and 3 more alleys along the midlines of the triangle.

10. Two people are playing a game, moving pieces on a plane. Player one controls a black piece ("wolf"); during his turn, he can move the wolf in any direction by not more than 1cm. Player two controls 100 white pieces ("sheep"); during his turn, he can move **one** of the sheep in any direction by not more than 1cm. Player one starts.

Is it true that no matter what the initial positions were, the wolf will be able to get at least one sheep?