MATH CLUB ASSIGNMENT 4: COMBINATORICS 2

OCT 22, 2023

1. How many "words" of length 12 can you form using just 2 letters, A and B, if each letter must appear 6 times? What if you are allowed to use 3 letters, A, B, C, each appearing 4 times? Can you get a general formula for number of words using 3 letters, appearing k_1, k_2, k_3 times respectively (thus, total length is $n = k_1 + k_2 + k_3$)?

2. An ant moves along the real line, starting at the origin and each time moving one unit either to the left or to the right. He takes $2n$ steps and ends up again at the origin
   (a) Show that the number of such paths is equal to the constant term in the expression $(x + x^{-1})^{2n}$.
   (b) Show that this number is equal to $_{2n}C_n$.

3. An ant moves in the plane, starting at the origin and each time moving one unit to the left or to the right or up or down. He takes $2n$ steps and ends up again at the origin.
   (a) Show that the number of such paths is equal to the constant term in the expression
   $(x + x^{-1} + y + y^{-1})^{2n}$.
   *(b) Prove that this number is equal to $(_{2n}C_n)^2$. (Hint: rotate the plane 45 degrees. Then each ant's step moves him both horizontally and vertically.)

Stars and Bars

4. How many ways there are to arrange 12 books on 2 bookshelves (top and bottom one)? The order on each bookshelf matters; there are no restrictions on how many of the 12 books are on top shelf.

5. How many solutions does the equation $x_1 + x_2 + x_3 = 2023$ have if x_1, x_2, x_3 must be non-negative integers? what if we require them to be positive integers?

6. How many different monomials in 3 variables x, y, z of total degree n are there? in 4 variables?

7. How many different monomials in 3 variables x, y, z of total degree n are there if we additionally require that each variable appears with positive degree (i.e. we look for monomials $x^a y^b z^c$, $a > 0$, $b > 0$, $c > 0$, $a + b + c = n$).

8. How many ways there are to put 15 chairs in 4 rooms if every room must have at least one chair? (Chairs are all identical, chairs inside the room are not ordered.)

*9. How many ways there are to put 15 people in 4 rooms if every room must have at least one person? (People are all different, people inside the room are not ordered.)